A NEW INTERMITTENCY MODEL INCORPORATING THE CALMING EFFECT

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Synopsis: In this paper, a transitional intermittency model is presented, which incorporates the stabilising effect of the calmed region trailing behind the turbulent spot. In order to estimate the intermittency, the flow is allowed to break down into turbulent spots throughout the transition zone with equal probability. The calming effect is then added to this to derive an expression for the modified spot formation distribution. This expression is used to arrive at an intermittency distribution that can be used to model the boundary layer parameters in the transitional zone. Even though this formulation is mainly valid for constant pressure flows, it is shown to be applicable to flows with pressure gradient as well.

List of Symbols

- $g$: Spot formation rate per unit area per unit time
- $g_c$: Corrected spot formation rate per unit area per unit time
- $g_n$: Spot formation rate (per unit distance per unit time) in concentrated breakdown model
- $G$: Non-dimensional spot formation rate
- $G_c$: Corrected non-dimensional spot formation rate
- $G_N$: Non-dimensional spot formation rate in concentrated breakdown model
- $N$: ‘Crumble’ = $G_N \cdot R_\theta^3$
- $q$: Freestream turbulence level
**Introduction**

The study of laminar-turbulent transition in attached boundary layer flows is a problem of tremendous importance both from the fundamental and applied perspectives. This has led to a spate of studies of this problem ever since Emmons [1] discovered turbulent spots and characterised the transition zone as being made up of these constituent spots - islands of turbulence in a laminar sea.

The intermittency ($\gamma$), is defined as the fraction of time for which the flow is turbulent at a point in the flow in the transitional zone. It is one of the most important parameters to model the transitional boundary layer parameters. The turbulent spots, which are formed due to the breakdown of the laminar flow, are seen to grow bigger till they coalesce to form the fully turbulent region. Hence the intermittency factor varies gradually from zero to unity across the transition zone.

The distribution of intermittency depends on the spot formation distributions across the transition zone. Over the years, various proposals for the spot formation rate and intermittency distribution have been put forward. These vary from the so-called continuous breakdown hypothesis of Emmons [1] and its variants (e.g., [2]) to the concentrated breakdown hypothesis of Narasimha [3]. Opinion has been divided as to the appropriateness of these hypotheses and they have led to many debates, especially in the field of turbomachinery, where transition modelling remains to be one of the most pressing problems.

In this work, we seek to reconcile the different modelling philosophies in vogue and this is expected to be achieved by incorporating two new factors - the stabilising effect of the so-called calmed region and that of the fully turbulent fluid in inhibiting further spot production. In the following sections, after explaining the background of intermittency model in the transition zone, the effects of the calmed region and the turbulent spot in modifying the spot production distribution are quantified to arrive at an expression for intermittency distribution.

The analyses presented here are mainly valid for steady flows but the basic formulation has more general applicability for unsteady flows also, as shown by Schulte & Hodson [4]. The consideration of unsteady effects is especially important in LP turbine applications where wakes from the upstream blade rows makes the flow inherently unsteady and hence the transitional dynamics is quite different from the steady situation. However, as long as the time scale of unsteadiness of the mean flow is much larger than the time scale of the turbulence in
the transitional boundary layer, one can tackle this problem as a quasi-steady flow (as far as transition modelling is concerned). Hence steady transitional modelling ideas can be brought to bear on the problem at hand. A recent work on the transitional boundary layer measurements in axial compressors and turbines, is given by Halstead et al. [5].

2. Background

2.1 Bypass Transition

If we consider the boundary layer development over a flat plate, the initially laminar flow, through a sequence of instabilities, breaks down into turbulent spots. The turbulent spots grow and merge as they propagate downstream to eventually encapsulate the whole flow field downstream. In turbomachinery flows, where the freestream disturbance levels are quite high, most of the initial stages of instabilities could be completely bypassed. This bypass transition is primarily the transition route of concern to us here. Fig. 1 shows the situation when breakdown occurs at only one streamwise location. This is therefore an example of concentrated breakdown.

The prediction of the transition onset location, is a formidable (and open) problem in hydrodynamic stability. Hence, it is usual to describe it in terms of a critical Reynolds number based on the boundary layer momentum thickness. For typical LP turbine situations, the peak freestream turbulence level in the neighbourhood of transition is typically of the order of about 4% [6]. Under these circumstances, the corresponding momentum thickness based Reynolds number would be about 168 at the transition onset location according to the correlation of Mayle [7]. In many LP turbines, this means that the transition sometimes begins before peak suction on the suction surface, but almost always ends after peak suction in the adverse pressure gradient region. It is this aspect, viz., that transition begins and ends in very different regimes that has led to the debates regarding the breakdown hypotheses.

2.2 Turbulent spots

Fig. 2 shows the schematic of a turbulent spot in bypass transition in a two-dimensional constant pressure flow. As Schubauer and Klebanoff [8] found, the spots are roughly heart shaped in planform with a pointed leading edge; the spots are often approximated by a triangular planform shape (following [9]). In the elevation, the spot can be seen to have an overhang in its shape which travels with a celerity (velocity) of about 88% of the freestream velocity; the trailing edge of the spot travels with a celerity of about half that of the freestream velocity. The planform shape varies slowly across the height of the boundary layer such that the planform at the boundary layer edge becomes a point.

The difference in the leading and trailing edge celerities of the boundary layer means that the spot grows in spatial extent as it moves downstream. Kinematically, since the spot is seen to have a self-similar form, this would mean a constant angle of propagation of the wedge enveloping the spot. This angle is about 21° for constant pressure flows.

Schubauer and Klebanoff [8] also found that a region of non-turbulent calmed region trails behind the turbulent spot. This calmed region is characterised by fuller velocity profiles than the laminar boundary layer which makes it less susceptible to destabilisation. Recent experiments (e.g., [10]) indicate that the calmed region is nearly rectangular in planform and its trailing edge is seen to travel with about 25% of the freestream velocity.
2.3 Intermittency Models

Following [1], consider the spatio-temporal evolution of a turbulent spot in planform. Together with the streamwise \((x)\) and spanwise \((z)\) co-ordinates, by considering the evolution in time \((t)\), the three-dimensional \(x-z-t\) space is constructed. A point \(P(x,z,t)\) in this space gives the relative location in space and time of different points in the planform of a turbulent spot. A spot generated at \(P(x, z, t)\) will sweep out a volume called the propagation volume in this space; a slice of this volume at a constant value of time \((t)\) would reveal the planform of the spot at that instant. By extrapolating the generators of the propagation volume upstream of point \(P\), we can construct the so-called dependence volume for the point \(P\). Only spots which are created within this volume can make the point \(P\) turbulent.

The dependence volume of \(P(x,z,t)\) is shown in Fig. 3. Here, the spot planform is taken to be triangular for the sake of simplicity following [9]. By working out the probability that a spot will be formed in the propagation volume, Emmons[1] derived the expression for intermittency at any point \(x\) as

\[
\gamma = 1 - \exp\left\{-\iiint g(x,z,t) dV\right\},
\]

where \(g(x,z,t)\) is the spot formation rate per unit time per unit area and \(dV\) is an elemental volume in the propagation cone; the product \(g dV\) is the probability that a spot is formed in the volume \(dV\). Note that, even though the equation (2.1) is valid even when \(g(x, y, t)\) is non-stationary in all the variables, we will consider here only steady two-dimensional transition so that \(g\) is only a function of \(x\). Further, if the spot propagation wedge is linear, as it is for constant pressure flows, it can be shown ([1], [11]) that

\[
\iiint dz_0 dt_0 = \frac{\sigma}{U} (x_0 - x)^2,
\]

where

\[
\sigma = U \iint_S \frac{dx dz}{x^3},
\]

is the non-dimensional spot propagation parameter and it is equal to 0.25 for flat plate flow [11]; the integration is carried out over the spot planform \((S)\). Since the integrand in equation (2.3) is weighted by \(1/x^3\), it is biased more towards the trailing edge (or the lower limit of integration). Hence,

\[
\gamma = 1 - \exp\left\{-\frac{\sigma}{U} \int_{x_0}^{x} \frac{2(x_0 - x)^2 dx_0}{x^3}\right\}.
\]

As a first approximation, Emmons assumed \(g\) to be a constant i.e., the spots are assumed to be formed with equal probability everywhere. This ‘continuous breakdown’ hypothesis leads to the expression
\[ y = 1 - \exp\left\{ -\frac{\sigma}{3U} g(x - x_t)^3 \right\} \]  

for intermittency. Here \( x_t \) is the transition onset location from which the integration is started.

Schubauer and Klebanoff [8] measured intermittency distributions on a flat plate and they found that Emmons’ hypothesis of ‘continuous breakdown’ did not agree with their measurements. To resolve this, Narasimha [3] proposed that the spots were born in a narrow belt around the onset location, and the length of this belt is so small in comparison to the transition zone length that the spots could be assumed to be formed only at the location \( x_t \). This could be modelled by a Dirac Delta function centered at \( x_t \), such that this ‘concentrated breakdown’ hypothesis would lead to

\[ g = g_n \quad x = x_t \]  
\[ g = 0 \quad x \neq x_t \]  

\[ y = 1 - \exp\left\{ -\frac{\sigma}{U} g_n(x - x_t)^2 \right\} \]

The spot formation rate in Narasimha’s model \( (g_n) \) has the dimensions of no. of spots per unit distance per unit time; whereas \( g \) in Emmons’ model has the dimensions of no. of spots/area/time.

Further, by defining \( \lambda = x(\gamma = 0.75) - x(\gamma = 0.25) \), which is a measure of the transition zone extent, and the non-dimensional streamwise distance \( \xi = (x - x_t)/\lambda \) in equation (2.6b), the ‘universal intermittency distribution’

\[ y = 1 - \exp\left\{ -0.41\xi^2 \right\} \]

is obtained. This expression has been found to be in very good agreement with measurements in constant pressure flows.

Even though the universal intermittency distribution (2.7) has been mainly derived for zero pressure gradient flows, it is found to be a good description of many pressure gradient flows as well [12]. However, there has been a lot of debate, especially in the field of turbomachinery where strong adverse and favourable pressure gradients are encountered at high freestream disturbance levels, as to the appropriateness of this hypothesis. There have even been suggestions that a hypothesis like Emmons’ ‘continuous breakdown’ hypothesis is more relevant for bypass transition. (e.g., [2],[7],[13]).

Originally, it was suggested by Emmons that during continuous breakdown, the spot formation rate should increase downstream. It was thought that with the increase in Reynolds number, the flow becomes more unstable. However, the mean velocity profile becomes fuller, and hence more stable, downstream as the intermittency rises. This would imply a decreasing spot formation rate with increasing \( x \). In reality, the breakdown of the laminar flow is neither ‘continuous’ nor ‘concentrated’ strictly but it is somewhere in between these two extreme pictures. In practice, for steady constant pressure flows, the belt over which the breakdown occurs is so small in comparison to the length of the transition zone, that the breakdown could
be safely assumed to be concentrated at a preferred streamwise location for purposes of modelling.

It is only in recent years that the importance of the calmed region in transition zone has begun to be realised (see [4]). Since the calmed region is a very stable region with a full velocity profile, it was suggested by Schulte & Hodson [4] that the birth of spots is inhibited in these regions; also, a spot cannot be formed in a region which is already turbulent. These factors are now believed to explain the success of the ‘concentrated breakdown hypothesis’ in describing two-dimensional constant pressure steady flows. A reconciliation is hence effected between the ‘concentrated’ and ‘continuous’ breakdown hypotheses by incorporating the role of calming and the turbulent regions.

In this paper, following [4], the calming effect and the role of turbulent spots in inhibiting further spot production are quantified, to arrive at expressions for spot formation distributions and a closed form expression for the intermittency distribution.

3. Transitional Intermittency with modified spot formation distribution

3.1 Propagation parameter for the spot and the calmed zone

In this section, the calmed region is quantified by a non-dimensional spot propagation parameter for use in later analyses. To do this, consider the turbulent spot with the calmed region trailing behind it (Fig. 2). The calmed region is seen to be fairly approximated by a rectangle in the planform. Then, following [1], we can define a propagation parameter ($\sigma''$) for the calmed region. This is given by (cf. Equation (2.3))

$$\sigma'' = Ut \int \frac{wdx}{C^3},$$

where $w$ is the width of the calmed region and the integration is from the spot trailing edge to the trailing edge of the calmed region ($C$). For the half angle of the propagation wedge $\alpha=10.5^\circ$, it can be shown that $\sigma'' = 1.11$.

The propagation parameter for the spot and the calmed region is the sum $\sigma' = \sigma + \sigma'' = 1.36$. With this, we can define another parameter ($\tau$)

$$\tau = \frac{\sigma'}{\sigma} = 5.44$$

3.2 Spot formation with calming and turbulence

In the present intermittency model, the spot formation rate is modified by the calming effect and the already turbulent fluid. To indicate this, we define the corrected spot formation rate by $g_c$. Note that this quantity is, in effect, a modified form of the spot formation rate $g$ used in Emmons’ model. Then the corrected intermittency is given by [4]

$$\gamma = 1 - \exp\left\{ - \int\!\!\!\int\!\!\!\!\int_{V} g_c(x_0, z_0, t_0) dV_0 \right\},$$

3.3
where \( dV_0 = dx_0 \, dz_0 \, dt_0 \). Physically, the expression (3.3) gives the probability that a flow is turbulent at a point since the integration is over the dependence volume \( V \) of the spot.

Since new spots cannot be born in a region which is already turbulent, or in a calmed region of an earlier spot (since the calmed region stabilises the flow and inhibits any flow breakdown), the probability of spot formation in an elemental volume \( dV \) is given by

\[
P(S) = g_c \, dV = P(S/L) \, P(L) = g \, dV \, P(L),
\]

In the above expression, \( P(L) \), is the probability that the flow is laminar, and \( P(S/L) = g \) is the probability that the spot is formed provided the flow is laminar (this is similar to the spot formation rate used in Emmons’ theory i.e., independent of any calming effect etc.). Here, \( g = g(x) \), i.e., the uncorrected spot formation rate can be, in general, a function of \( x \). For constant value of \( g(x) = g \), this is given by

\[
g_c(x) = g(x) \exp \left\{ - \int_{V_0}^{V} g_c(x_0) \, dx_0 \, dz_0 \, dt_0 \right\}.
\]

Since the propagators of the dependence volume for the spot as well as the calmed region are linear for a constant pressure flow, we could write

\[
\int S \int dV_0 \, dt_0 = \frac{\sigma}{U} (x_0 - x)^2.
\]

Hence, equation (3.4) becomes

\[
g_c(x) = g(x) \exp \left\{ - \frac{\sigma}{U} \int_{x_0}^{x} g_c(x_0) (x_0 - x)^2 \, dx_0 \right\}.
\]

This expression means that the corrected spot production rate at any location is obtained by starting with an uncorrected value at the onset location and as we march downstream allowing the calming and the turbulent zone effects to modify the spot formation rate. Notice, from equation (3.5), that its integrand vanishes as \( x_0 \to x \). This implies that the corrected spot formation rate at any location \( x \) is influenced only the corresponding values upstream of \( x \), but not by its value at \( x \) itself.

Equation (3.5) can be solved analytically to get a closed form solution. To do this, substitute the functional form

\[
g_c(x) = g(x) \varphi(x)
\]

in equation (3.5). This leads to (after rearranging)
\[-\ln[\phi(x)] = \int_{x_0}^{x} \frac{\sigma}{U} g(x_0) \phi(x_0) (x_0 - x)^2 \, dx_0\]

By inspecting both the sides of this expression, we can immediately see that
\[
\frac{d}{dx_0} \left[ \int_{x_0}^{x} \frac{1}{\phi(x_0)} \right] = \frac{\sigma'}{U} g(x_0) (x_0 - x)^2,
\]
and by integrating the last expression and rearranging,
\[
\phi(x_0) = \frac{1}{\int_{x_0}^{x} \frac{\sigma}{U} g(x_0) (x_0 - x)^2 \, dx_0 + C}
\]
where $C$ is an arbitrary constant which becomes unity by virtue of the initial condition $\phi(x_0 = x_t) = 1$. Hence
\[
\phi(x_0) = \frac{1}{1 + \int_{x_0}^{x} \frac{\sigma}{U} g(x_0) (x_0 - x)^2 \, dx_0}
\]
so that, for $g(x) = g = \text{constant}$ (which is considered here for the sake of simplicity)
\[
\phi(x_0) \approx \frac{1}{1 + \frac{\sigma}{U} g \left( (x - x_t)^3 - (x - x_0)^3 \right)}
\]
Hence, the corrected spot formation rate at any streamwise location $x$ can be written as
\[
g_c(x) = \frac{g}{1 + \frac{g \sigma}{3U} (x - x_t)^3}
\]

Some observations are in order with regard to the expression (3.6). The factor $\sigma'$ in (3.6) denotes the effect of both the calmed region and the already turbulent spot in modifying the spot production rate. Accordingly, if we put $\sigma' \to 0$ in this expression, we get $g_c \to g$, i.e., the spots are allowed to breakdown continuously as in Emmons’ model. To go to the other limit, if we allow $\sigma' = (\sigma + \sigma'' \to \infty$ by allowing $\sigma''$ to go to $\infty$, then
\[
g_c(x) = g \quad \text{at} \quad x = x_t
\]
\[
g_c(x) = 0 \quad \text{at} \quad x \neq x_t
\]
The first condition (at $x = x_t$) comes about because $(x - x_t)^3$ goes to zero faster than $\sigma'$ goes to infinity and hence the denominator in (3.6) becomes unity. This is same as the concentrated breakdown model (equation 2.6a).

It should be noted that, even if we do not include calming in the model, and take $\sigma' = \sigma$, the denominator in the expression (3.6) is greater than unity (for $x > x_t$). This means that, even without the calming effect, the breakdown rate is a decreasing function of $x$, and hence different from the continuous breakdown picture. This is due to the inhibition of further spot production in an already turbulent region. On top of this, if we also take calming into account, the breakdown rate tends more and more towards the concentrated breakdown model as the amount of calming is increased.

A schematic of the spot formation rate with the concentrated and continuous breakdown models are shown along with that given by the present model in Fig. 4.

An estimate of the streamwise extent over which the spots are formed can be calculated as follows. If we take the location at which the spot formation rate drops to 10% of the initial value as the last location of spot birth (denoted by $x_s$), then from (3.6) it can be seen that

$$x_s - x_t = 3 \left( \frac{U}{g\sigma'} \right)^{1/3}. \quad 3.7$$

This expression gives the belt over which spots are born. This belt will shrink to zero width (as in the concentrated breakdown model) if either $g$ or $\sigma'$ is large or if $U$ is small. This belt is about half the length of the transition zone for constant pressure flows (this estimate is arrived at by making use of the relation (3.11) between the spot formation rates of concentrated breakdown and the present theory as used in the next section).

### 3.3 A new intermittency distribution

The expression (3.6) can be cast in terms of non-dimensional variables as

$$G_c = \frac{G}{1 + \tau G(X - X_t)^3}, \quad 3.8$$

where

$$G_c = \frac{g_c\sigma v^3}{3U^4}, \quad X = \frac{U}{v}, \quad G = \frac{g\sigma v^3}{3U^4} \quad \text{and} \quad X_t = \frac{U x_t}{v}.$$

$\tau = \sigma'/\sigma$, as in the previous sections. Since the non-dimensional spot formation rates (denoted by $G$ etc.,) are likely to be microscopic in value, these non-dimensional $G$’s by themselves have no physical significance. They are used here for the sake of convenience as they have been used by other workers (e.g., [2]) in the past. In terms of these non-dimensional variables, the expression for intermittency becomes
\[
\gamma = 1 - \exp\left\{- \frac{X}{X_r} \int \frac{3G_c}{X_0 - X} \left( X_0 - X \right)^2 dX_0 \right\},
\]

which, by using equation (3.8) becomes

\[
\gamma = 1 - \exp\left\{- \frac{X}{X_r} \int \frac{3G}{1 + \tau G \left( X_0 - X \right) \left( X_0 - X \right)^2} dX_0 \right\}.
\]

This integral can be evaluated to give the final intermittency variation (after some algebra) and this expression is shown in Appendix A.

As mentioned earlier, the concentrated breakdown model has been found to be an excellent description of transition in constant pressure steady flows and it has been validated against different experiments over the years. Hence, in order to validate the new intermittency distribution, we compare it with the concentrated breakdown model. In order to do this, we need a relation between the spot formation rates at the onsets for both the theories, i.e., between \( g \) and \( g_n \), or between the non-dimensional variables \( G \) and \( G_n \) (see Appendix B). Here, the concentrated and the present models are matched such that they give the same intermittency over most of the transition zone. This gives

\[
G = 0.45G_n^{3/2},
\]

Also a relation is needed between the transition onset locations in both the models as these locations could be expected to be quite different from physical considerations. Here, we find that

\[
X_t(Conc) - X_t(present) = 27,000.
\]

The relation (3.12) above has been found numerically for constant pressure flows. Numerical experiments with different pressure gradients indicate that this constant pressure situation is likely to be an upper bound for difference between onset location (according to concentrated breakdown and the present theory).

Using expressions (A1, 3.11 and 3.12), the variation of intermittency with \( X = \frac{Ux}{v} \) is shown plotted in Fig. 5, where it is compared with the concentrated breakdown model. It can be seen that the agreement between these models is excellent thereby validating the current model for constant pressure steady flows.

One of the main things to be noted in Fig. 5, apart from the excellent agreement between these theories over most of the transition zone, is that the present model has a faster pickup close to the origin than the concentrated breakdown model. Close to the onset, many experiments have in fact shown that the intermittency is non-zero even upstream of the onset location inferred by the concentrated breakdown model. This bears out the trend shown by the present model. This is a clear demonstration that the effect of calming and the turbulent fluid are crucial factors in determining the intermittency distribution realistically in constant
pressure flows and more so in pressure gradient flows where the calming/turbulent factor could be expected to have strong influence on the intermittency distribution [4].

3.4 Pressure gradient flows

For pressure gradient flows, the calculation of intermittency is performed numerically as follows. The spot planform is assumed to be triangular (as in [9]) and the calmed region is taken to be of rectangular planform. The momentum thickness $\theta$ is calculated by the Thwaites method and from this the Polhausen pressure gradient parameter $\lambda \theta = \frac{\theta^2}{\nu} \frac{dU}{dx}$, is obtained. Gostelow et al. [14] have correlated the spot formation rate $G_n$ as a function of the pressure gradient parameter $\lambda \theta$ at the transition onset location and the freestream turbulence level $q$. The correlation for $G_n$ is given by

$$G_n = 0.86 \times 10^{-3} \exp \left\{ 2.134 \lambda \theta \ln(q) - 59.23 \lambda \theta - 0.5641 \ln(q) \right\}.$$

The corresponding $G$ is obtained by using a modified form of the relation (3.11) and this is given by

$$G = G_N^{3/2}.$$

The change in the constant of proportionality is required because of the assumed triangular planform for the spot which reduces $\sigma$ from 0.25 to 0.17. Then by using equations (3.3-3.4), the intermittency expression is obtained by numerical integration. This calculation has been performed for different pressure gradients as shown in Fig. 6. The leading edge, trailing edge and calmed region celerities are 88%, 50% and 25% of the freestream velocity respectively. The Reynolds number for the calculation is typical of a low pressure turbine.

Fig. 6 presents a comparison of calculated steady flow intermittency distributions. The two breakdown models are compared for a range of pressure distributions. In each case, transition begins at $\text{Re}_{\theta,\text{trans}} = 168$ which corresponds to an inlet turbulence intensity of approximately 4 percent. It can be seen that in mildly accelerating and decelerating flows ($\lambda \theta = -0.04,-0.04$), the agreement is very good. In a strong adverse pressure gradient ($\lambda \theta = -0.08$), which is most relevant to low pressure turbines and compressors, the agreement is excellent. In a zero pressure gradient flow, the form of the curves is identical but there is a displacement of the distributions. This was noted earlier. In practice, this displacement could be allowed for. However, it amounts to no more than about 10 percent of the length of the transition zone. On an LP turbine blade, transition often starts near peak suction and probably occurs over approximately 10-15 percent of the surface length. In practice, therefore, the differences between the two models are relatively small in all circumstances. Therefore, it is concluded that the present model may be used in situations where the pressure gradient changes significantly within the transition zone. Such a situation arises in many low pressure turbines.

4. Conclusions

A new transitional intermittency model incorporating the calming effect is presented. The spot formation rate at any location is allowed to be modified by the calmed region as well as the already turbulent fluid. The intermittency distribution obtained from this model is
compared with the concentrated breakdown and in doing so correlations between the spot formation parameters between these models are obtained for constant pressure flow. By numerical experiments for different pressure gradients, the validity of this correlation for pressure gradient flows has also been demonstrated.

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References
Appendix A

The solution to equation (3.10) is

\[ \gamma = 1 - \exp\{-I\} \]

where

\[ I = \frac{1}{\tau} \ln\left\{ 1 + \tau G(X - X_f)^3 \right\} + 3G(X - X_f)I_1 \]

\[ I_1 = \frac{A}{2(\tau G)^{2/3}} \ln\left\{ (\tau G)^{2/3}(X - X_f)^2 - (\tau G)^{1/3}(X - X_f) + 1 \right\} + \]

\[ \frac{2}{\sqrt{3}(\tau G)^{1/3}} \left( \frac{A}{2(\tau G)^{1/3}} + B \right) \left\{ \tan^{-1} \left[ \frac{2(\tau G)^{1/3}(X - X_f) - 1}{\sqrt{3}} \right] + \frac{\pi}{6} \right\} + \frac{C}{(\tau G)^{1/3}} \ln\left\{ (\tau G)^{1/3}(X - X_f) + 1 \right\} \]

and

\[ A = -\frac{(X - X_f)(\tau G)^{1/3} + 2}{3} , \]

\[ B = \frac{2}{3} \left[ (X - X_f) - (\tau G)^{1/3} \right] , \]

\[ C = \frac{(X - X_f) + 2(\tau G)^{-1/3}}{3} . \]

(A1)

Appendix B

The objective is to obtain a functional form of the relation between \( G \) and \( G_n \). To do this, we match the \( \lambda \)'s obtained from the concentrated and continuous breakdown theory (without any calming). This is done as the analysis is tractable this way. The relation between \( G \) and \( G_n \) obtained this way is likely to be different from the present model including calming but the functional form of the relation is nevertheless the same. Hence by matching \( \lambda \)'s between equations (2.5) and (2.6b), we get

\[ \frac{0.641}{G_n^{1/2}} = \frac{0.4545}{G^{1/3}} . \]

This leads to the functional form
If we include calming also, then for the present theory $k = 0.45$ is seen to give good agreement with the concentrated breakdown model. The general form of the above expression is consistent with the fact that the intermittency expressions for the concentrated and continuous breakdown models vary as $\exp(-x^2)$ and $\exp(-x^3)$ respectively (see equations (2.5) & (2.6b)).
Fig. 1 Schematic of bypass transition

Fig. 2 Schematic of a turbulent spot
Fig. 3 Dependence volume of a turbulent spot and its calmed region

Fig. 4 Schematic of spot formation rates according to different breakdown models
Fig. 5 Intermittency distribution for a constant pressure flow

Fig. 6 Intermittency distributions for pressure gradient flows