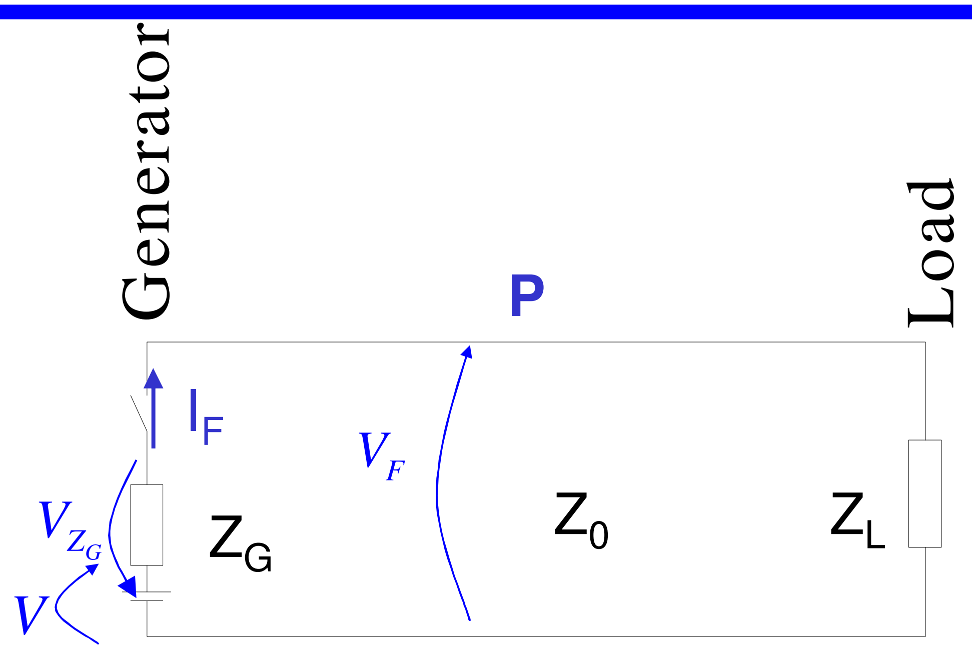
I.3.5 Ringing

Ringing=Unwanted oscillations of voltage and/or current

Ringing is caused by **multiple reflections**. The original wave is reflected at the load, this reflection then gets reflected back at the generator, etc, etc

We will illustrate this by looking at the step change in voltage when a device is switched on



1.At switch-on, a pulse V_F is generated and travels towards the load

At the generator:

$$V - V_{Z_G} = V_F \quad \text{and} \quad I_F Z_G = V_{Z_G}$$

Thus: $V_F = V - I_F Z_G$

But, for a unidirectional (un-reflected) wave: $I_F = \dfrac{V_F}{Z_0}$

$$\Rightarrow \quad V_F = V - \frac{V_F}{Z_0} Z_G \quad \Rightarrow \quad V_F = \frac{Z_0}{Z_0 + Z_G} V$$

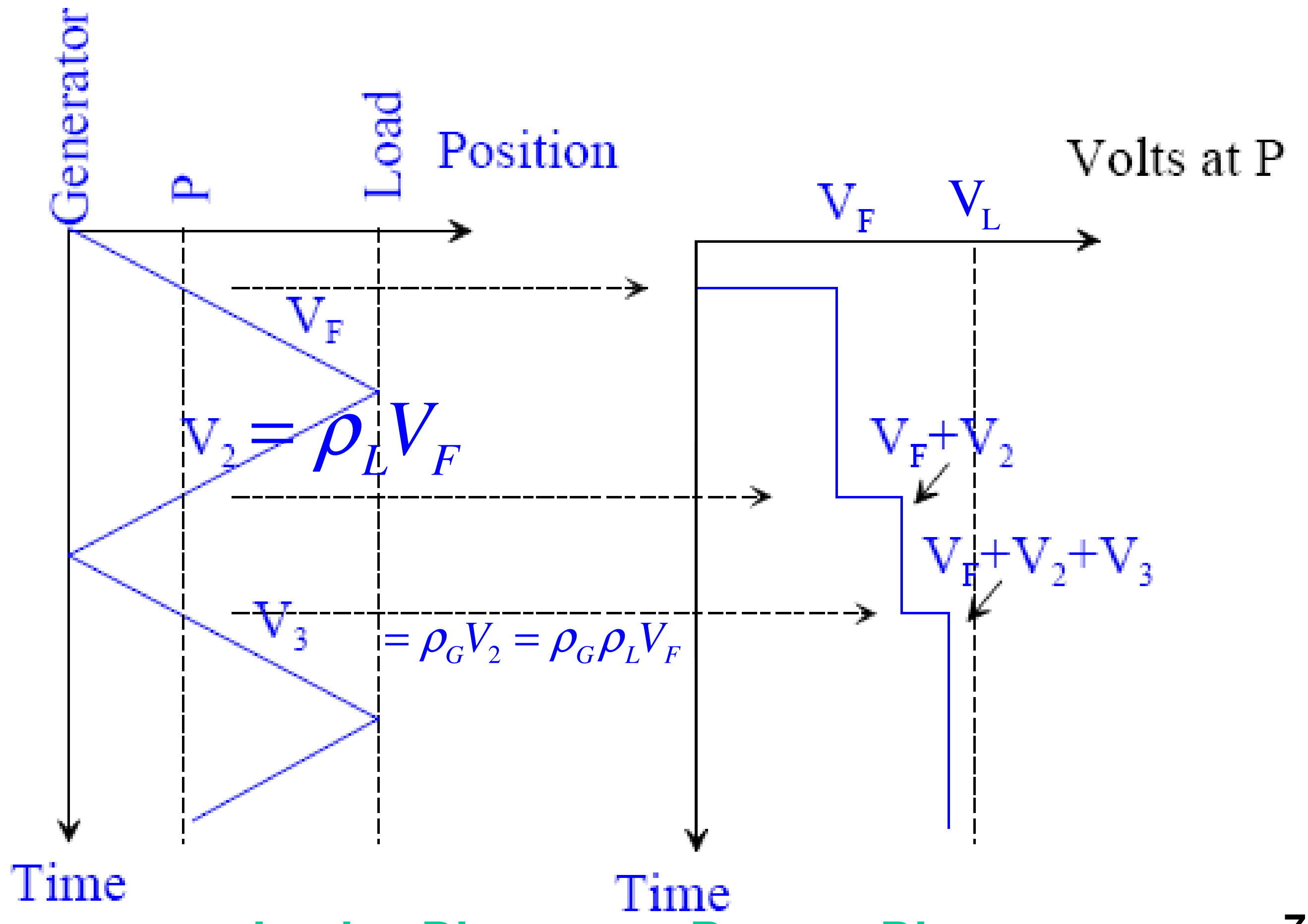
2. Part of the pulse is then reflected at the load as $V_B = V_2 = \rho_L V_F$

Where $$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

3. V_2 is reflected at the generator as $V_3 = \rho_G V_2 = \rho_G \rho_L V_F$

Where $$\rho_G = \frac{Z_G - Z_0}{Z_G + Z_0}$$

4. The amplitude at the load asymptotically approaches V_L



Lattice Diagram or Bounce Diagram

What is the asymptotic value?

$$V_P^1 = V_F$$

$$V_P^2 = V_F + \rho_L V_F = V_F(1 + \rho_L)$$

$$V_P^3 = V_F + \rho_L V_F + \rho_G \rho_L V_F = V_F(1 + \rho_L + \rho_G \rho_L)$$

$$V_P^4 = V_F + \rho_L V_F + \rho_G \rho_L V_F + \rho_G \rho^2{}_L V_F =$$

$$V_F(1 + \rho_L + \rho_G \rho_L + \rho_G \rho_L{}^2)$$

...

$$V_P^n = V_F(1 + \rho_L + \rho_G \rho_L + \rho_G \rho_L{}^2 + \rho_G{}^2 \rho_L{}^2 + \rho_G{}^2 \rho_L{}^3 \ldots)$$

V_P^n can be rewritten as:

$$V_P^n = V_F[1 + \rho_G \rho_L + (\rho_G \rho_L)^2 + \ldots] + V_F \rho_L [1 + \rho_G \rho_L + (\rho_G \rho_L)^2 + \ldots]$$

Hence, the asymptotic value, for n⇒∞ is:

$$V_P^\infty = V_F[1 + \rho_L] \sum_0^\infty (\rho_G \rho_L)^n$$

Since, by definition, $|\rho_L| \leq 1$ and $|\rho_G| \leq 1$

and for $|x| \leq 1$ $\sum_0^\infty x^n = \frac{1}{1 - x}$

$$\Rightarrow \quad V_P^\infty = V_F[1 + \rho_L] \frac{1}{1 - \rho_G \rho_L}$$

Substituting the definitions of ρ_G and ρ_L:

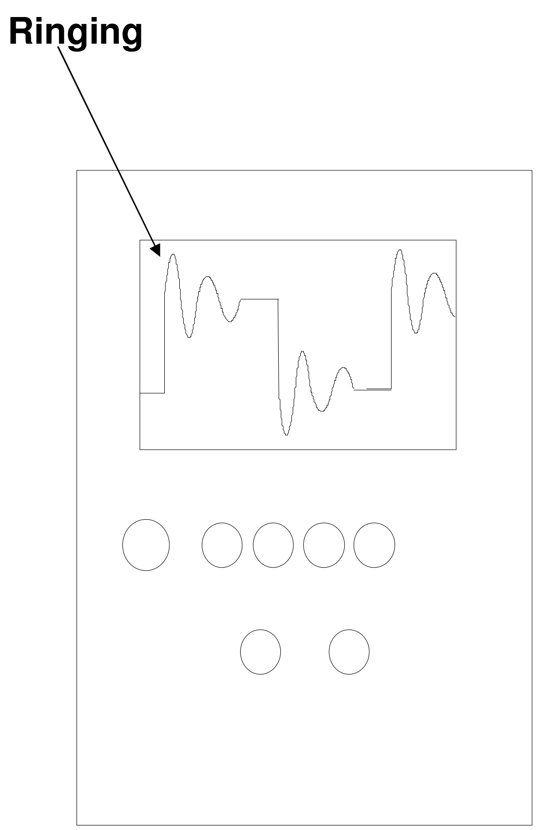
$$V_P^{\infty} = \frac{Z_L}{Z_L + Z_G} V = V_L$$

Thus, if we wait long enough, any "transmission line" effects should go away, and we converge to what we would have if the line was just some wire connecting the source to the load

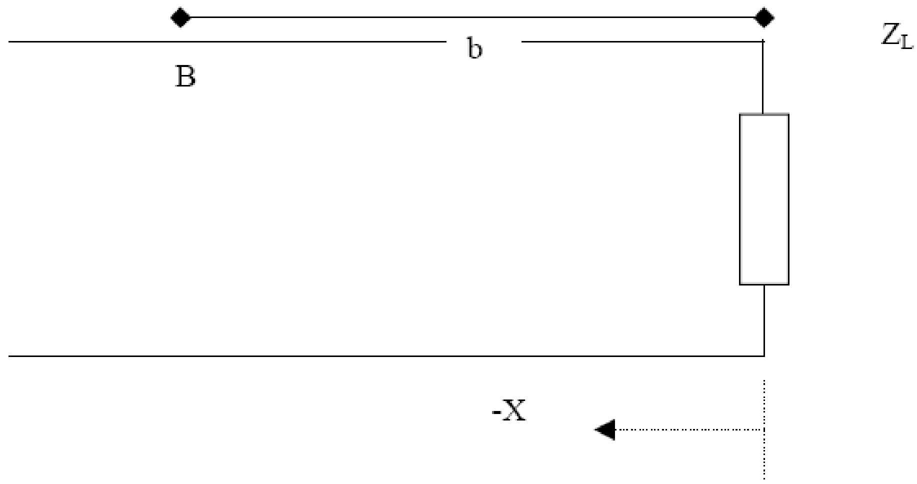
In this case, the load resistor and the source resistor would form a voltage divider⇒ the voltage across the load is determined by the voltage divider equation

If $Z_L \rightarrow \infty$, open circuit, then:

$$V_P^{\infty} = V$$



I.3.6 ¼ Wave Matching



The impedance of a line is Z_0 only in the absence of reflections. With reflections the impedance at point B is a function of:

- Intrinsic impedance Z_0
- Impedance of the load Z_L
- Distance from the load
- Wavelength.

The general expression of impedance at x is: $\overline{Z}(x) = \dfrac{\overline{V}(x)}{\overline{I}(x)}$

Remembering that: $\overline{V}(x) = \overline{V}_F e^{-j\beta x} + \overline{V}_B e^{j\beta x}$

$$\overline{I}(x) = \overline{I}_F e^{-j\beta x} + \overline{I}_B e^{j\beta x}$$

$$\overline{I}_F = \frac{\overline{V}_F}{Z_0} \qquad \overline{I}_B = -\frac{\overline{V}_B}{Z_0}$$

Then:

$$\overline{Z}(x) = \frac{\overline{V}(x)}{\overline{I}(x)} = \frac{\overline{V}_F e^{-j\beta x} + \overline{V}_B e^{j\beta x}}{\dfrac{\overline{V}_F}{Z_0} e^{-j\beta x} - \dfrac{\overline{V}_B}{Z_0} e^{j\beta x}} = Z_0 \frac{e^{-j\beta x} + \dfrac{\overline{V}_B}{\overline{V}_F} e^{j\beta x}}{e^{-j\beta x} - \dfrac{\overline{V}_B}{\overline{V}_F} e^{j\beta x}}$$

Since, from (3.1a), (3.1b): $\overline{\rho_L} = \frac{\overline{\overline{V_B}}}{\overline{\overline{V_F}}} = \frac{\overline{\overline{Z_L}} - Z_0}{\overline{\overline{Z_L}} + Z_0}$

We get:

$$\overline{Z}(x) = Z_0 \frac{e^{-j\beta x} + \overline{\rho_L} e^{j\beta x}}{e^{-j\beta x} - \overline{\rho_L} e^{j\beta x}} = Z_0 \frac{(\overline{Z_L} + Z_0)e^{-j\beta x} + (\overline{Z_L} - Z_0)e^{j\beta x}}{(\overline{Z_L} + Z_0)e^{-j\beta x} - (\overline{Z_L} - Z_0)e^{j\beta x}}$$

Remembering that:

$$e^{+j\beta x} = \cos(\beta x) + j\sin(\beta x)$$
$$e^{-j\beta x} = \cos(\beta x) - j\sin(\beta x)$$

and cos(-x)=cos(x) sin(-x)=-sin(x)

We can replace the exponential with sin and cos and substitute x=-b

$$\overline{Z}_b = \overline{Z}(-b) = Z_0 \frac{\overline{Z}_L \cos(\beta b) + jZ_0 \sin(\beta b)}{Z_0 \cos(\beta b) + j\overline{Z}_L \sin(\beta b)}$$

$$\overline{Z}_b = \overline{Z}(-b) = Z_0 \frac{\overline{Z}_L + jZ_0 \tan(\beta b)}{Z_0 + j\overline{Z}_L \tan(\beta b)} \quad (3.3)$$

A quarter of a wavelength back from the load $b = \frac{\lambda}{4}$

Remembering that: $\beta = \frac{2\pi}{\lambda}$ ➡ $\beta b = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$

We get $\overline{Z}_b = Z_0 \frac{\overline{Z}_L + jZ_0 \tan(\pi/2)}{Z_0 + j\overline{Z}_L \tan(\pi/2)}$

Since $\tan(\pi/2) = \infty$

The impedance at point $b = \frac{\lambda}{4}$ is:

$$\overline{Z}_b = \frac{Z_0^2}{\overline{Z}_L} \qquad (3.4)$$

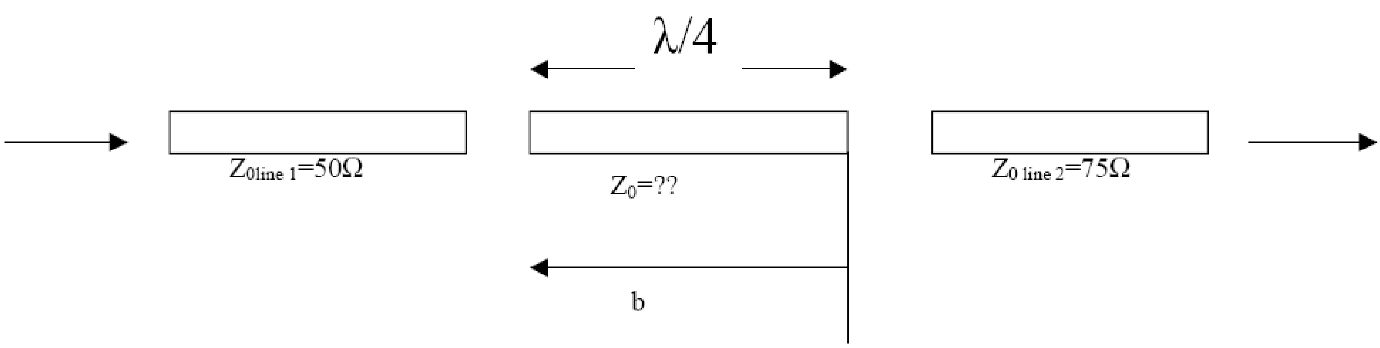
This expression is important when we are trying to connect two lines with different impedances, and we do not want to have any reflections

This leads to the concept of quarter wave transformer

I.3.7 Quarter wave transformer

Two lines are to be linked. The first has an impedance Z_{0Line1}= 50 Ω, while the second has an impedance Z_{0Line2}= 75 Ω

What should the impedance Z_0 of a quarter wavelength section of line be, in order to eliminate reflections?



Second line appears as Z_L= Z_{0Line2}=75Ω to the ¼ wave link

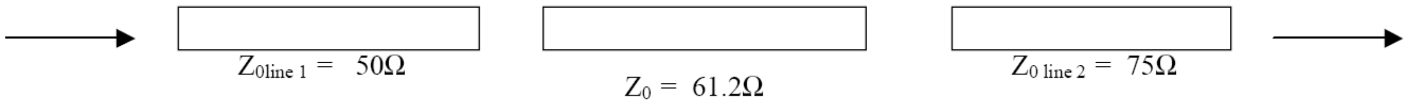
Remembering that

$$\overline{\rho_L} = \frac{\overline{V_B}}{\overline{V_F}} = \frac{\overline{Z_L} - Z_0}{\overline{Z_L} + Z_0}$$

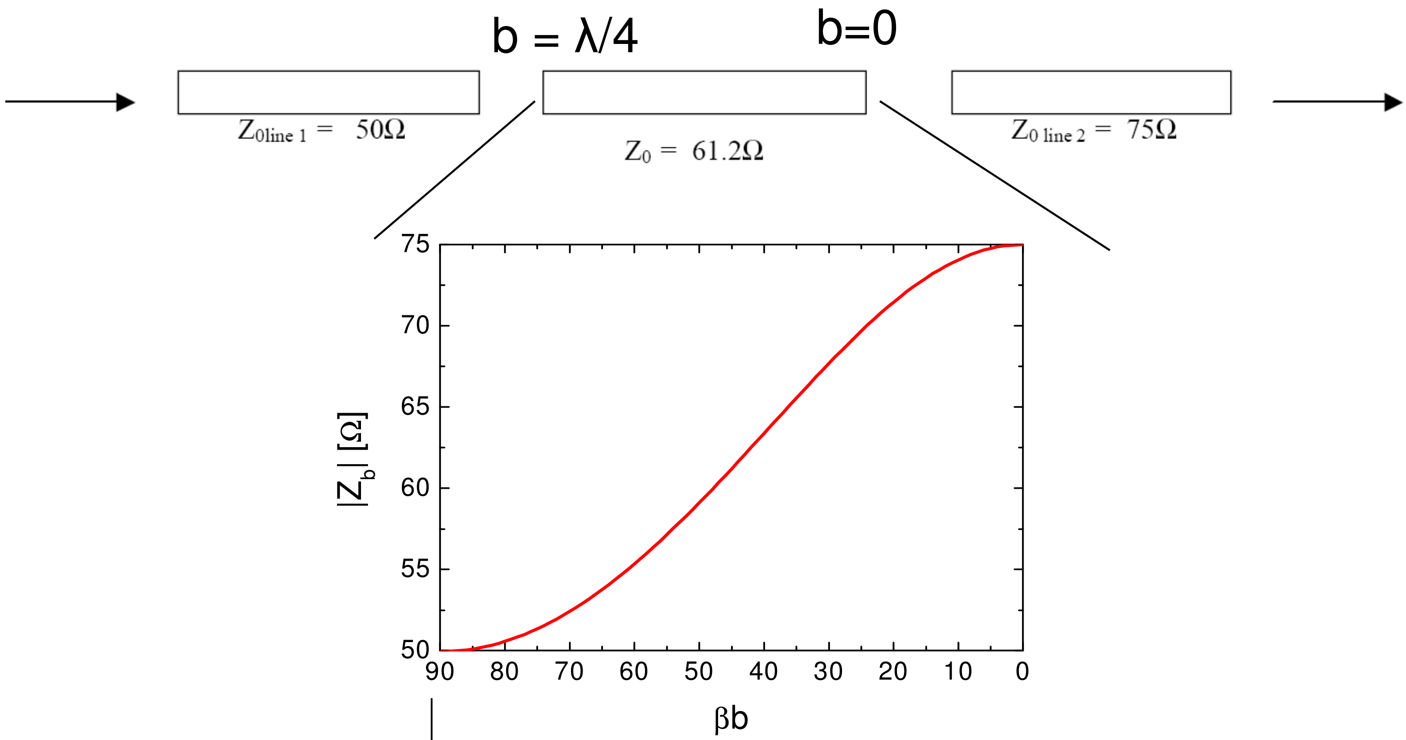
To have no reflections we need $\overline{\rho_L} = 0 \Rightarrow \overline{Z_L} = Z_0$

We want Z at b to equal 50Ω, the Z_0 of line 1, so there is no reflection back along the line. Hence

$$Z_{0Line1} = Z_b = \frac{Z_0^2}{Z_L} \quad \Rightarrow \quad 50 = \frac{Z_0^2}{75} \quad \Rightarrow \quad Z_0 = 61.2\Omega$$



The graph below shows how Z varies along the ¼ wavelength section. Note: this solution is only valid for one frequency



$$b = \frac{\lambda}{4} \Rightarrow \beta b = \frac{\pi}{2}$$

II.1 Electromagnetic Fields

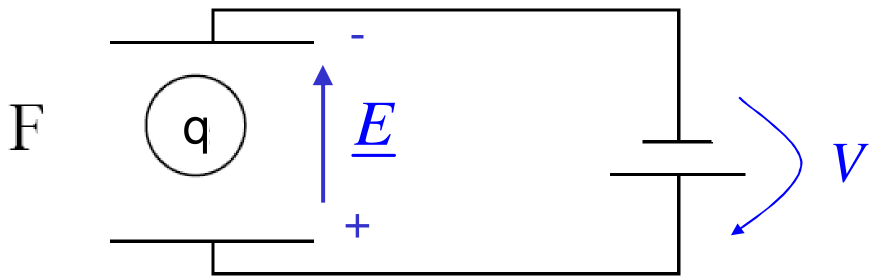
Aims

Define E, B, D, H, recall basic maths and Maxwell's equations

Objectives

At the end of this section you should be able to describe the relation between E, B, D, H, understand the meaning of displacement current density and its role in Maxwell's equations

II.1.1 Definitions



A charge q placed in an electric field experiences a force F which is dependent on the charge itself and on the electric field strength E

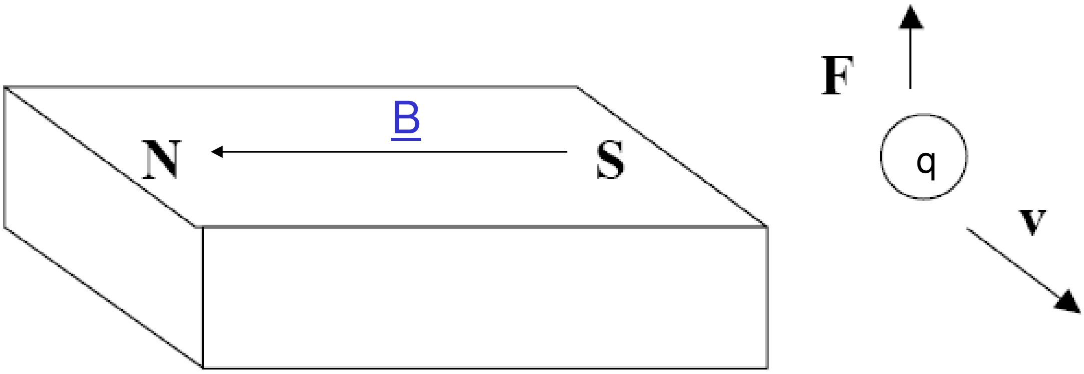
We define Electric Field, E , so that

$$\underline{F} = q\underline{E}$$

For an electron: q=-1.602 10^{-19} C

Units $[E] = \frac{N}{C} = \frac{V}{m}$

Magnetic Flux



A charge moving with velocity $\underline{v}$ in a magnetic field also experiences a force

We define Magnetic Flux Density, $\underline{B}$, such that

$$\underline{F} = q\underline{v} \times \underline{B} \qquad \text{Lorentz Force}$$

For an electron: q=-1.602 10^{-19} C

Units $[B] = \frac{N}{C}\frac{s}{m} = \frac{N}{mA} = T \quad (Tesla)$

Electric and magnetic fields are closely related. One can give rise to the other, and vice versa

Electric fields are not only created by charges (such as the charge on the plates of a capacitor)

but also by a **changing magnetic field**

Magnetic fields are created not only by moving charges, i.e. current in a coil or aligned spins in an atom (as in a permanent magnet)

but also by changing **electric fields**
(Maxwell's displacement current, as discussed later)

In addition to the above, we have to allow for charges and currents in materials. We thus define two new quantities:

Electric Flux Density: D [D]=C/m^2

Magnetic Field Intensity: H [H]=A/m

In linear materials, D and E; B and H, are directly related by the permittivity ε and permeability μ of such materials

$$D=\varepsilon E \qquad B=\mu H$$

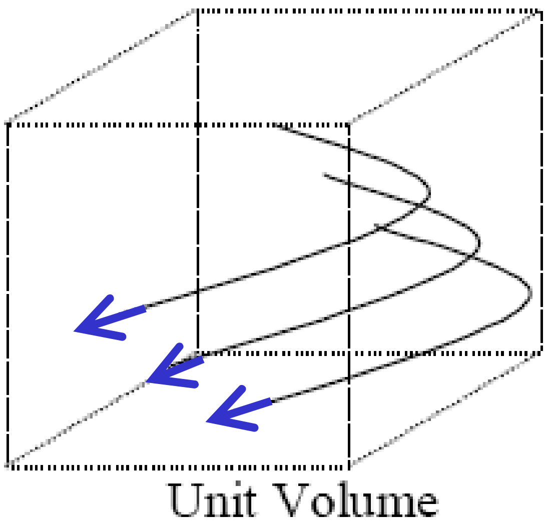
Permittivities and permeabilities are often expressed relative to those of free space:

$\varepsilon=\varepsilon_0\varepsilon_r$, where ε_0 is the permittivity of free space

$\mu=\mu_0\mu_r$, where μ_0 is the permeabilitiy of free space

and ε_0= 8.854 x 10^{-12} F/m μ_0= 4π 10^{-7} H/m

Representation of Flux Density B

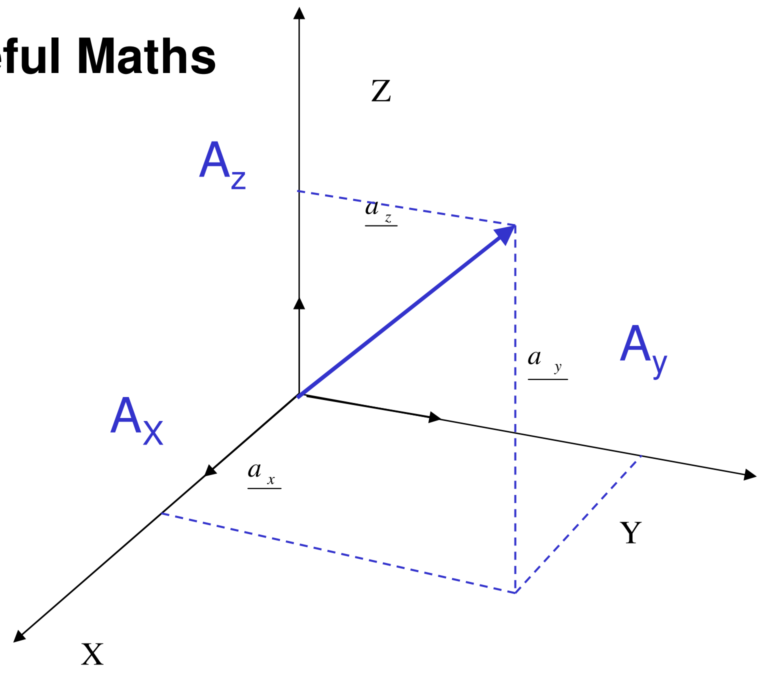


Strength of the field = **Number** of flux lines per unit area

Direction of the field = **Direction** of the flux lines

II.1.2 The laws of electromagnetism

II.1.2.0-Useful Maths



Vector Notation

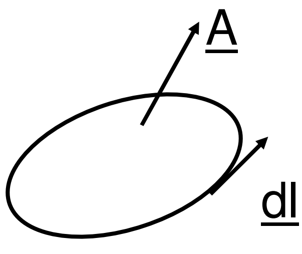
$$\underline{A} = \mathbf{A} = A_x \underline{a_x} + A_y \underline{a_y} + A_z \underline{a_z}$$

Curl $\underline{A}$: $\underline{\nabla} \times \underline{A} =$

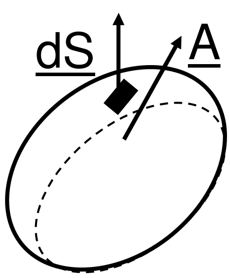
$$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\underline{a_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\underline{a_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\underline{a_z}$$

Divergence of $\underline{A}$ (div $\underline{A}$)

$$\underline{\nabla} \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$



$$\oint_c \underline{A}.d\underline{l} = \text{circulation integral}$$



$$\int_s \underline{A}.d\underline{S} = flux$$

Stokes Theorem

$$\oint_c \underline{A}.d\underline{l} = \int_S (\nabla \times \underline{A}).d\underline{S}$$

Divergence Theorem

$$\int_S \underline{A} \cdot d\underline{S} = \int_V \nabla \cdot \underline{A} dV$$

II.1.2.1 Maxwell's Laws

Integral Form

Maxwell-Faraday: $\oint_c \underline{E}.d\underline{l} = -\int_s \dot{\underline{B}}.d\underline{S}$

Maxwell-Ampere: $\oint_c \underline{H}.d\underline{l} = \int_s \left(\underline{J} + \dot{\underline{D}}\right).d\underline{S}$

where

$$\dot{\underline{B}} = \frac{\partial \underline{B}}{\partial t}$$

$\underline{J}$ Conduction current density [A/m^2]

$\dot{\underline{D}} = \frac{\partial \underline{D}}{\partial t}$ Displacement current density [A/m^2]

Differential Form

$$\nabla \times \underline{E} = -\dot{\underline{B}}$$

$$\nabla \times \underline{H} = \underline{J} + \dot{\underline{D}}$$

Differential form linked to integral form by Stokes theorem

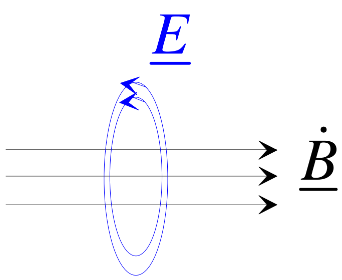
$$\oint_c \underline{E}.d\underline{l} = \int_S (\nabla \times \underline{E}).d\underline{S}$$ Stokes

but $$\oint_c \underline{E}.d\underline{l} = -\int_s \dot{\underline{B}}.d\underline{S}$$ Maxwell Faraday

⇒ $$\nabla \times \underline{E} = -\dot{\underline{B}}$$

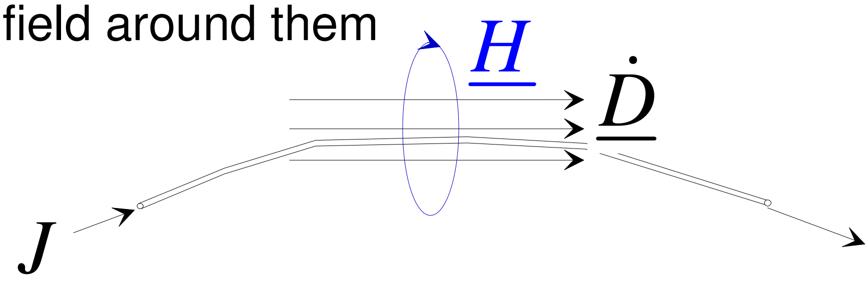
The Maxwell-Faraday Law $\oint_c \underline{E}.d\underline{l} = -\int_s \dot{\underline{B}}.d\underline{S}$ implies that

a changing magnetic flux $\dot{\underline{B}}$ has rings of field $\underline{E}$ around it



The Maxwell-Ampere Law $\oint_c \underline{H}.d\underline{l} = \int_s \left(\underline{J} + \dot{\underline{D}}\right).d\underline{S}$

implies that a steady current $\underline{J}$ or a changing electric flux $\dot{\underline{D}}$ have rings of H field around them



II.1.2.2 Gauss's Laws

Differential Form

$$\nabla \cdot \underline{B} = 0$$

$$\nabla \cdot \underline{D} = \rho$$ where ρ charge density [C/m³]

Integral Form

$$\int_S \underline{B}.d\underline{S} = 0$$

$$\int_S \underline{D}.d\underline{S} = Q$$ where Q total charge [C]

We can get the integral form from the differential one, by integrating over volume and applying the divergence theorem

For example $\int_V \nabla \cdot \underline{B} dV = \int_S \underline{B} \cdot d\underline{S}$ Divergence

But $\nabla \cdot \underline{B} = 0$ ⇒ $\int_S \underline{B} \cdot d\underline{S} = 0$

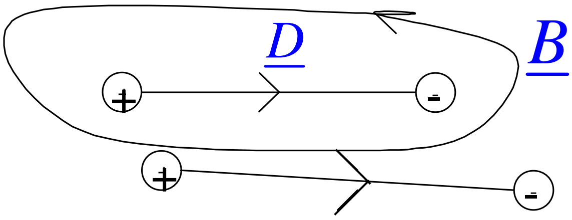
The divergence ∇ gives a measure of the difference between the number of flux lines entering a volume and those leaving it:

$\nabla \cdot \underline{X}$ is positive $\qquad \nabla \cdot \underline{X} = 0 \qquad \nabla \cdot \underline{X}$ is negative

Hence, Gauss' laws imply that:

1)The B flux lines are continuous, i.e. they are never broken. A flux line exiting from the north pole of a magnet will return to that magnet at the south pole

2) The D flux lines are continuous except when broken by point charges ⇒ lines of D begin and end on point charges



II.1.2.3 Displacement current density

It is one of Maxwell's key contributions and explains electromagnetic waves propagation

In Ampere's law the term $\dot{\underline{D}}$ is missing. Thus:

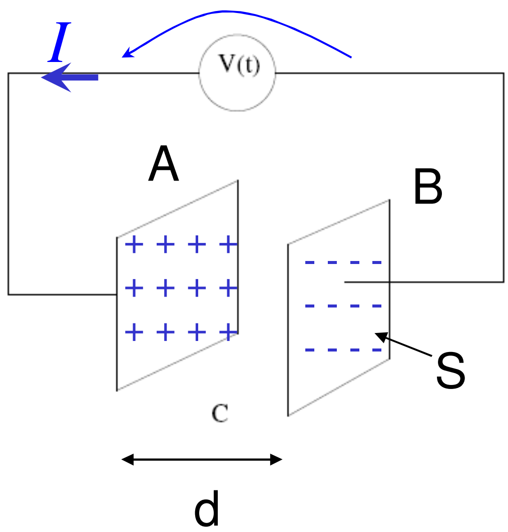
$$\oint_c \underline{H}.d\underline{l} = \int_s \left(\underline{J}\right).d\underline{S} = I$$

This applies in a wire

Maxwell added the term $\dot{\underline{D}}$ to take into account situations such as a wire with a break in it carrying an a.c. current

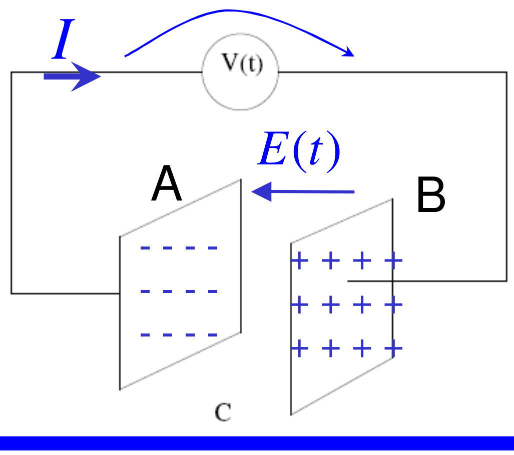
The displacement current allows us to take into account the effect of the gap formed by the break

Consider a capacitor with an applied voltage V(t)



The charges on plate A move to B, thus creating a current I(t)

When the voltage reverses



However, the charges of opposite sign on the two plates create a varying electric field E(t), thus a varying electric field density $D(t)=\varepsilon_0 E(t)$

$$\frac{\partial D}{\partial t} = \frac{\partial(\varepsilon_0 E)}{\partial t} = \frac{\partial(\varepsilon_0 V / d)}{\partial t} = \frac{\varepsilon_0}{d}\frac{\partial V}{\partial t}$$

But, in a parallel plate capacitor $C = \frac{\varepsilon_0 S}{d}$ and $I = C\frac{\partial V}{\partial t}$

➡ I(t) in the circuit= $C\frac{\partial V}{\partial t} = \frac{\varepsilon_0}{d}\frac{\partial V}{\partial t} S = \frac{\partial D}{\partial t} S$

Hence: $\frac{\partial D}{\partial t} = \frac{I}{S} = J_D$ Displacement current density

II.2 Electromagnetic Waves

Aims

To derive the equations for wave propagation in free space, as well as dielectric and conducting media

Objectives

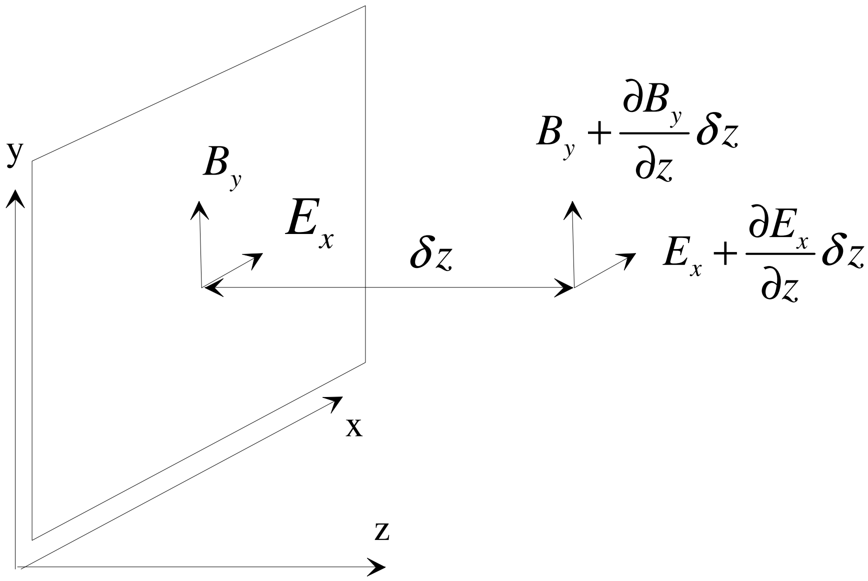
At the end of this section you should be able to describe the propagation of a plane wave, derive its velocity and intrinsic impedance in any medium

II.2.1 Derivation of Wave Equation

Consider an infinite plane z = 0 in which, at all points

$\underline{E} = \left(E_x, 0, 0\right) e^{j\omega t}$ and $\underline{B} = \left(0, B_y, 0\right) e^{j\omega t}$

Hence $\underline{E}$ and $\underline{B}$ are perpendicular and uniform



In the plane $z = \delta z$, the fields will have varied by the rates of change of B and E with z

We can use the diagram to evaluate Maxwell's equations and derive the wave equation

$$\oint_c \underline{E}.d\underline{l} = -\int_s \dot{\underline{B}}.d\underline{S}$$ Maxwell-Faraday

$$\left(E_x + \frac{\partial E_x}{\partial z}\delta z\right)\delta x + 0 + \left(-E_x \delta x\right) + 0 = -\left(\frac{\partial B_y}{\partial t} + \frac{\partial^2 B_y}{\partial z \partial t}\delta z\right)\delta z \delta x$$

We ignore second order terms. Hence we have $\frac{\partial B_y}{\partial t}$ and not $\frac{\partial^2 B_y}{\partial t \partial z}$

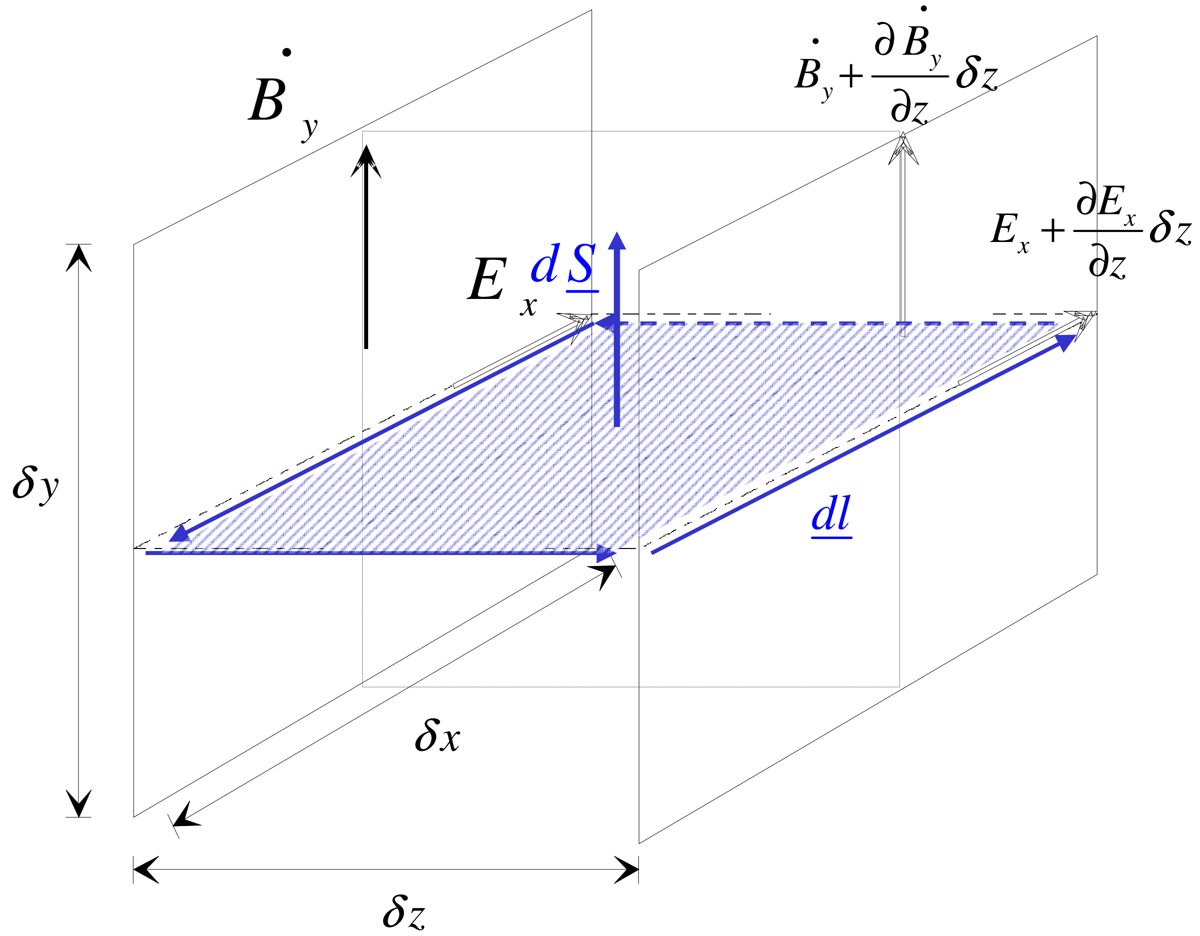
$$\Rightarrow \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

Note that the differential form is:

$\underline{\nabla} \times \underline{E} = -\dot{\underline{B}}$

$$\Rightarrow \frac{\partial E_x}{\partial z} = -\dot{B}_y = -\frac{\partial B_y}{\partial t}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\underline{a_x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\underline{a_y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\underline{a_z} = -\left(\dot{B}_x \underline{a_x} + \dot{B}_y \underline{a_y} + \dot{B}_z \underline{a_z}\right)$$



If we then repeat the calculation, but this time using $\underline{H}$ and $\underline{\dot{D}}$ and taking the plane $\delta z \delta y$

No currents

From $\oint_c \underline{H}.d\underline{l} = \int_s (\underline{J} + \underline{\dot{D}}).d\underline{S}$ (Maxwell-Ampere)

$$\Rightarrow -\left(H_y + \frac{\partial H_y}{\partial z}\delta z\right)\delta y + H_y \delta y = \frac{\partial D_x}{\partial t}\delta z \delta y$$

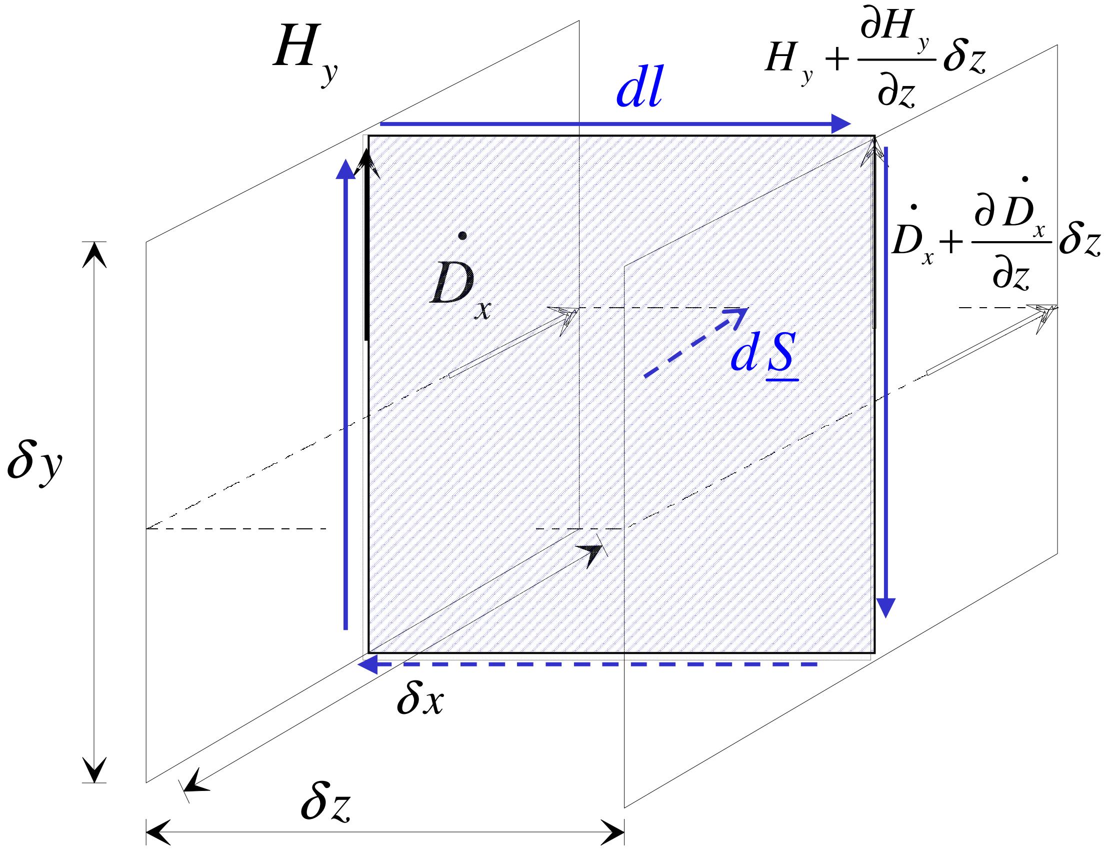
$$\Rightarrow \frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t}$$

Note that the differential form is:

$$\Rightarrow -\frac{\partial H_y}{\partial z} = \dot{D}_x = \frac{\partial D_x}{\partial t}$$

$$\underline{\nabla} \times \underline{H} = \underline{J} + \underline{\dot{D}}$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\underline{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\underline{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\underline{a}_z = \left(\dot{D}_x \underline{a}_x + \dot{D}_y \underline{a}_y + \dot{D}_z \underline{a}_z\right)$$



B and H are directly related by the permeability μ: $\underline{B} = \mu \underline{H}$

D and E are directly related by the permittivity ε: $\underline{D} = \varepsilon \underline{E}$

Summarising:

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad \frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t}$$

The next step is to eliminate B from the first equation and D from the second

Since $B = \mu H$ $\quad D = \varepsilon E$

We get the following equations in E and H:

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad (5.1)$$

$$\frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t} \qquad (5.2)$$

These are similar to the telegrapher's equations:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \qquad (1.1)$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \qquad (1.2)$$

Applying the same technique of differentiating eq. 5.1 and 5.2 with respect to z, and substituting in from 5.2, we end up with the equations for electromagnetic waves in free space (or pure dielectric medium, with J=0)

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu\frac{\partial}{\partial t}\left(\frac{\partial H_y}{\partial z}\right) = \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 H_y}{\partial z^2} = -\varepsilon\frac{\partial}{\partial t}\left(\frac{\partial E_x}{\partial z}\right) = \mu\varepsilon\frac{\partial^2 H_y}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 H_y}{\partial z^2} = \mu\varepsilon\frac{\partial^2 H_y}{\partial t^2}$$

These have the same form as the equations for waves in transmission lines. Therefore they have similar solutions

Wave velocity is defined by:

$velocity = \frac{1}{\sqrt{\mu\varepsilon}}$ analogous to $\frac{1}{\sqrt{LC}}$

In free space $\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{8.854 \cdot 10^{-12} \left[\frac{F}{m}\right] \cdot 4\pi \cdot 10^{-7} \left[\frac{H}{m}\right]}} \approx 3 \cdot 10^8 \frac{m}{s}$

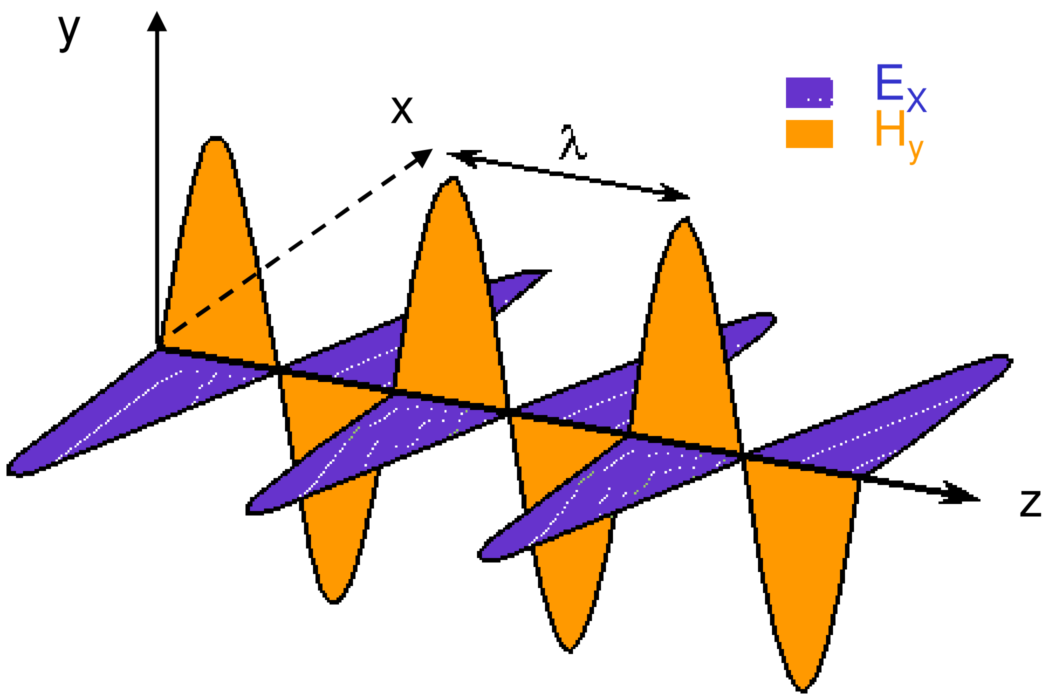
Remembering that $[F] = \frac{s}{\Omega}$ and $[H] = \Omega \cdot s$

$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 3 \cdot 10^8 \frac{m}{s}$ speed of light

All the results obtained from the telegrapher's equations can now be reused for electromagnetic waves:

$$E_x = \mathbb{R}e\left\{\left(\overline{E_{xF}} e^{-j\beta z} + \overline{E_{xB}} e^{j\beta z}\right) e^{j\omega t}\right\}$$

$$H_y = \mathbb{R}e\left\{\left(\overline{H_{yF}} e^{-j\beta z} + \overline{H_{yB}} e^{j\beta z}\right) e^{j\omega t}\right\}$$



THE

LONDON, EDINBURGH AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FOURTH SERIES.]

MARCH 1861.

XXV. *On Physical Lines of Force.* *By* J. C. MAXWELL, *Pro-*

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$$\left.\begin{aligned} V &= E, \\ &= 310{,}740{,}000{,}000 \text{ millimetres per second}, \\ &= 193{,}088 \text{ miles per second.} \end{aligned}\right\} \quad . \quad (136)$$

The velocity of light in air, as determined by M. Fizeau*, is 70,843 leagues per second (25 leagues to a degree) which gives

$$\begin{aligned} V &= 314{,}858{,}000{,}000 \text{ millimetres} \\ &= 195{,}647 \text{ miles per second.} \quad . \quad . \quad . \quad . \quad (137) \end{aligned}$$

The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that *light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena.*

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II.2.2 Intrinsic Impedance

For electromagnetic waves we define the quantity:

η intrinsic impedance

This is a function of permeability and permittivity in the same way as Z_0, the characteristic impedance, was a function of inductance and capacitance per unit length

$$Z_0 = \sqrt{\frac{L}{C}} \qquad (2.1)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \qquad (5.3)$$

η links E & H in the same way as Z_0 linked V & I

$$\eta = \frac{\overline{E_{xF}}}{\overline{\overline{H_{yF}}}} = -\frac{\overline{E_{xB}}}{\overline{\overline{H_{yB}}}}$$

$$\text{Similar to } Z_0 = \frac{\overline{V_F}}{\overline{\overline{I_F}}} = -\frac{\overline{V_B}}{\overline{\overline{I_B}}}$$

Note: The vectors $\underline{E}$ and $\underline{H}$ are orthogonal to one another, hence the subscripts x and y in our expression for η

In free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\cdot 10^{-7}}{8.854\cdot 10^{-12}}} \approx 377\Omega$$

II.2.3 Wave propagation in conducting media

So far we have considered EM waves propagation in air or in a pure dielectric medium: σ = conductivity = 0

If $\sigma \neq 0$ then $\underline{J} \neq 0$ $\underline{J} = \sigma \underline{E}$

The Maxwell-Ampere law becomes:

$$\oint_c \underline{H}.d\underline{l} = \int_s \left(\underline{J} + \dot{\underline{D}}\right).d\underline{S} = \int_s \left(\sigma \underline{E} + \varepsilon \dot{\underline{E}}\right).d\underline{S}$$

This changes the wave equation to:

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\sigma \frac{\partial E_x}{\partial t} + \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$ Helmholtz equation

We then proceed in a similar fashion to what done in lossy transmission lines

For simplicity we assume

$$E_x = f(z)e^{j\omega t}$$

$$\frac{\partial E_x}{\partial t} = j\omega f(z)e^{j\omega t} = j\omega E_x$$

$$\frac{\partial^2 E_x}{\partial t^2} = (j\omega)^2 E_x$$

$$\frac{\partial^2 E_x}{\partial z^2} = j\omega\mu(\sigma + j\omega\varepsilon)E_x = \gamma^2 E_x$$

$$\gamma = (\alpha + j\beta) = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \qquad (5.4)$$

propagation constant

$$E_x = \mathrm{Re}\left\{\left[\overline{E}_{xF} e^{-(\alpha + j\beta)z} + \overline{E}_{xB}\, e^{(\alpha + j\beta)z}\right] e^{j\omega t}\right\} \quad (5.5)$$

$$H_y = \mathrm{Re}\left\{\left[\overline{H}_{yF} e^{-(\alpha + j\beta)z} + \overline{H}_{yB} e^{(\alpha + j\beta)z}\right] e^{j\omega t}\right\} \quad (5.6)$$

Then

$$\frac{\overline{E}_{xF}}{\overline{H}_{yF}} = \eta = -\frac{\overline{E}_{xB}}{\overline{H}_{yB}}$$

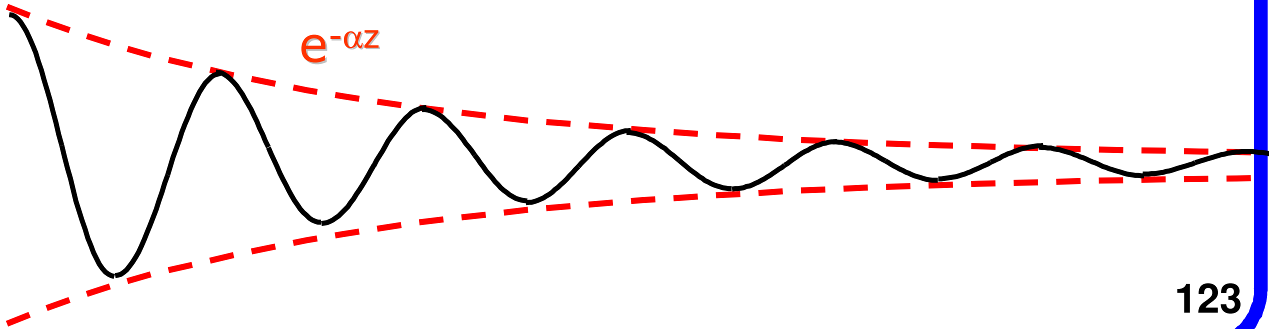
Where $\eta = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| e^{i\angle\overline{\eta}}$ (5.7)

is a complex impedance

Note that if $\sigma = 0$ (i.e. infinite resistivity) then

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

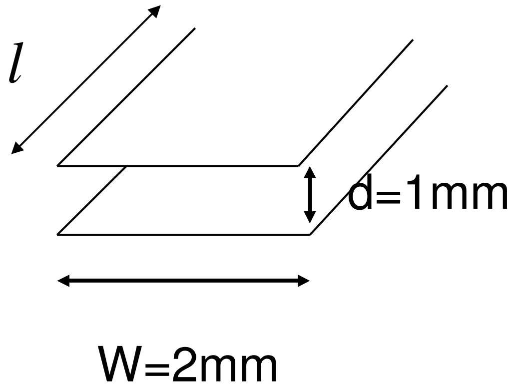
Having a finite conductivity leads to a complex impedance. This, in turn, leads to the E and H waves having a real decay term in the exponent. This signifies a wave with decaying amplitude as it travels into a conductive medium. The decay constant is α



II.2.4 Example – Characteristic Impedance

A printed circuit board is one millimetre thick, has an earthing plane on the bottom and $\varepsilon_r = 2.5$ & $\mu_r = 1$

Estimate the characteristic impedance of a track 2 mm wide



Capacitance $$C' = \varepsilon_0 \varepsilon_r \frac{Area}{d} = \varepsilon_0 \varepsilon_r \frac{Wl}{d}$$

Capacitance per unit length $$C = \frac{C'}{l} = \varepsilon_0 \varepsilon_r \frac{W}{d} \approx 44 \frac{pF}{m}$$

Wave velocity

$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r}}$$

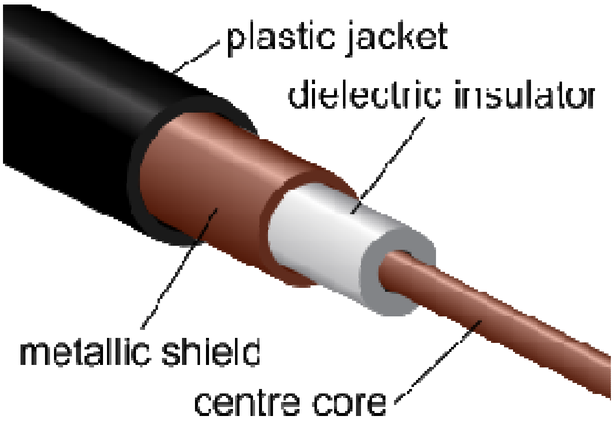
Hence :

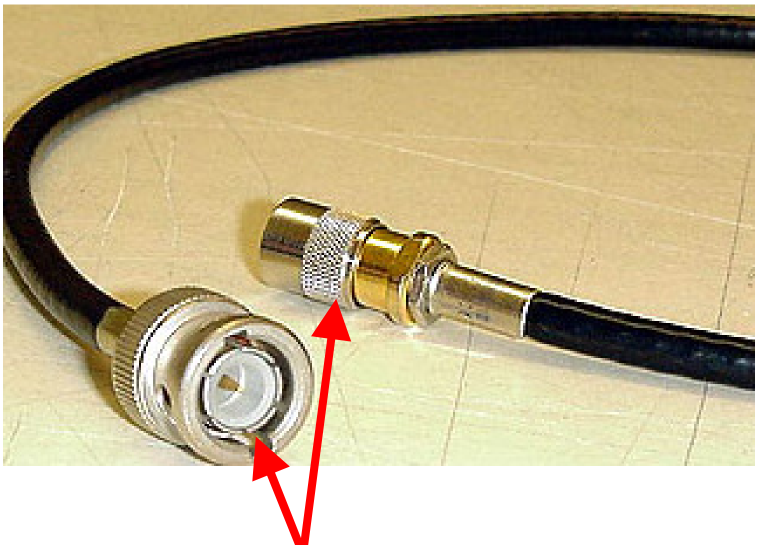
$$L = \frac{\varepsilon_0 \varepsilon_r \mu_0 \mu_r}{C} \approx 0.63 \cdot 10^{-6} \frac{H}{m}$$

Then

$$Z_0 = \sqrt{\frac{L}{C}} \approx 120\,\Omega$$

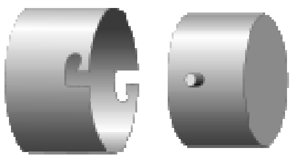
II.2.4 Example – Termination of a Coaxial Cable





BNC Connector
(Bayonet Neill Concelmann)

Bayonet mount locking



Patent 1951
Paul Neill
Carl Concelmann
Octavio Salati

The characteristic impedance is given by:

$$Z_0 = \sqrt{\frac{L}{C}}$$

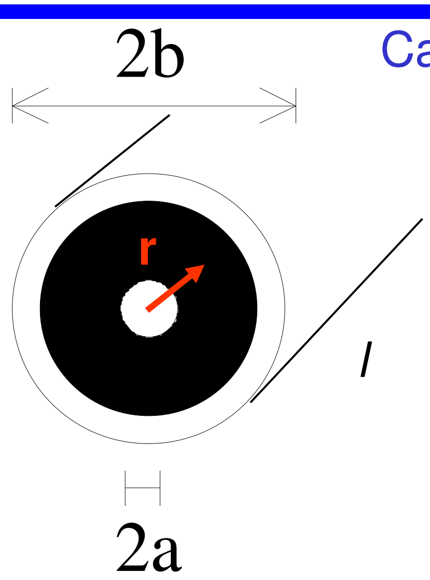
The voltage reflection coefficient is:

$$\overline{\rho}_L = \frac{\overline{Z_L} - Z_0}{\overline{Z_L} + Z_0}$$

In order to avoid unwanted reflections, we need a Z_L to terminate our coaxial cable with the same impedance as the characteristic impedance of the cable

$$Z_0 = \overline{Z_L}$$

Capacitance per unit length



b outer radius
a inner radius

$$C = \frac{q}{\Delta V} \quad \text{with } q = \frac{Q}{l} \quad \text{charge per unit lenght}$$

$$\int_S \underline{D}.d\underline{S} = Q \qquad \underline{D} = \varepsilon \underline{E}$$

$$\Rightarrow \quad \varepsilon \underline{E} 2\pi r l = q l$$

$$\Rightarrow \quad \underline{E} = \frac{q}{2\pi r \varepsilon}$$

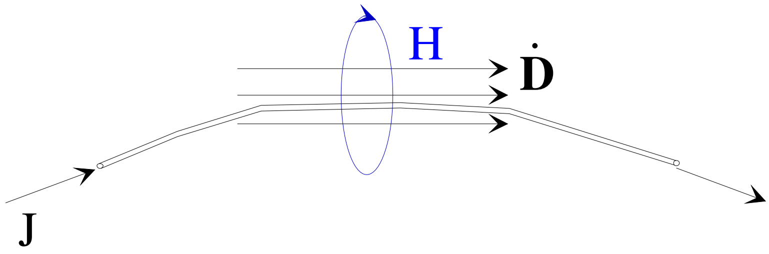
But $\underline{E} = -\frac{dV}{dr}$

Then

$$\Delta V = -\int_b^a \underline{E} dr = -\int_b^a \frac{q}{2\pi r \varepsilon} dr = -\frac{q}{2\pi\varepsilon} \ln\left(\frac{a}{b}\right) = \frac{q}{2\pi\varepsilon} \ln\left(\frac{b}{a}\right)$$

$$\Rightarrow \quad C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$

We can work out the inductance per unit length L of the cable from Ampere's law:



$$\oint_C \underline{H}.\underline{dl} = \int_S \left(\underline{J} + \underline{\dot{D}}\right).\underline{dS}$$

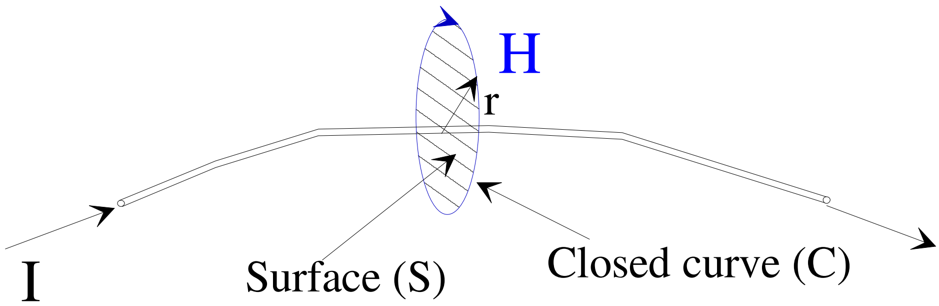
Where S is any surface bounded by a closed curve C

In the absence of a changing electric field $\underline{\dot{D}}$ we can simplify the equation to:

$$\oint_C \underline{H}.\underline{dl} = \int_s \left(\underline{J}\right).\underline{dS}$$

Since J is the current density then:

$$\int_S \underline{J}.\underline{dS} = I$$ for the surface shown below



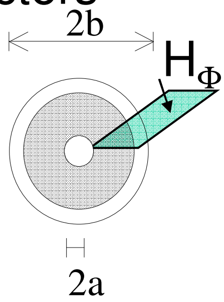
Note: since the field strength comes from the total current flowing, this explains how coaxial cables shield fields

The inner conductor carries current flowing in one direction. The outer in the opposite direction. Hence, summing the currents for a surface which includes both inner and outer conductors gives a total current of 0, thus a field strength of 0

The field strength per unit length in the cable is then:

$$H_\phi = \frac{I}{2\pi r}$$ Radial

The total magnetic flux per unit length, ψ, is derived by integrating the field strength per unit length between inner and outer conductors



dS=drdl **NOT 2πrdr**

$$\psi = \int_a^b \mu H_\phi dr = \frac{\mu I}{2\pi} \ln\left(\frac{b}{a}\right)$$

Since $$L = \frac{\psi}{I}$$

the inductance per unit length L is : $$L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

Hence

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)}{\frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}}} \Rightarrow Z_0 = \frac{\ln\left(\frac{b}{a}\right)}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}$$

If we consider a typical b/a~3-3.5

And a typical ε_r~2 (e.g. Teflon, PTFE)

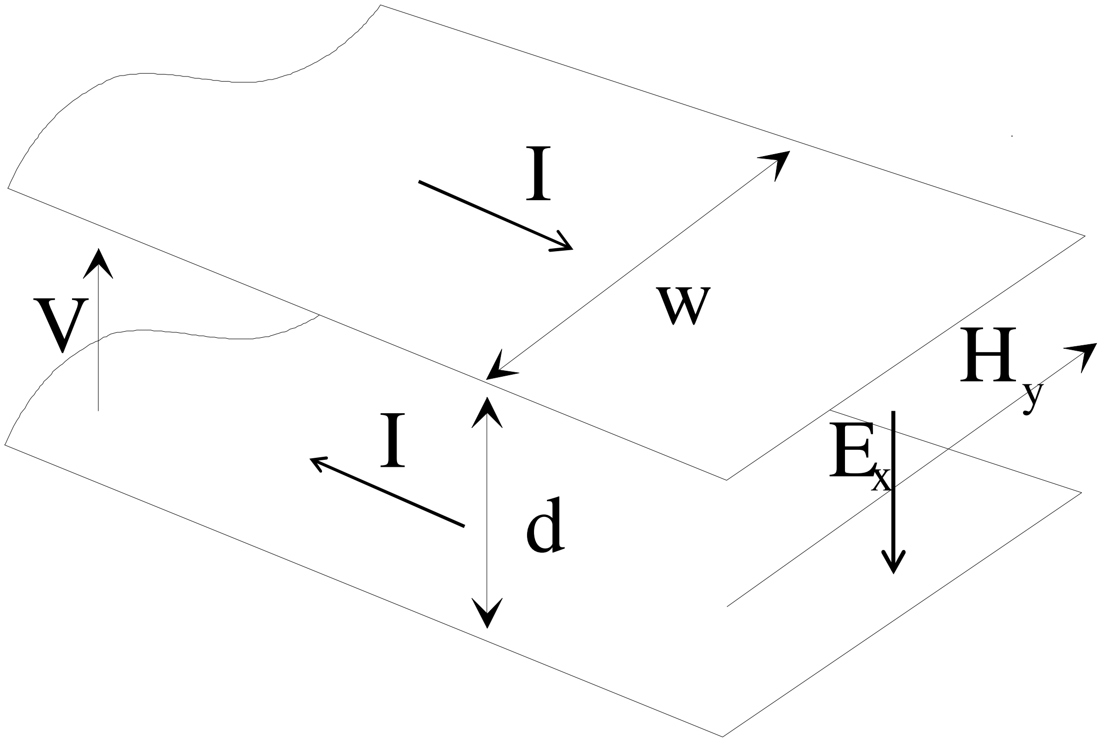
We get a typical value for Z_0 ~50Ω

Thus, we can use a 50 Ω resistor to terminate our coax

So, for example, if we have an input impedance of 50Ω for a television aerial socket, then the aerial lead should also have a characteristic impedance of 50Ω

II.2.6 The Poynting vector and power in EM waves

Let us consider a parallel plate transmission line:



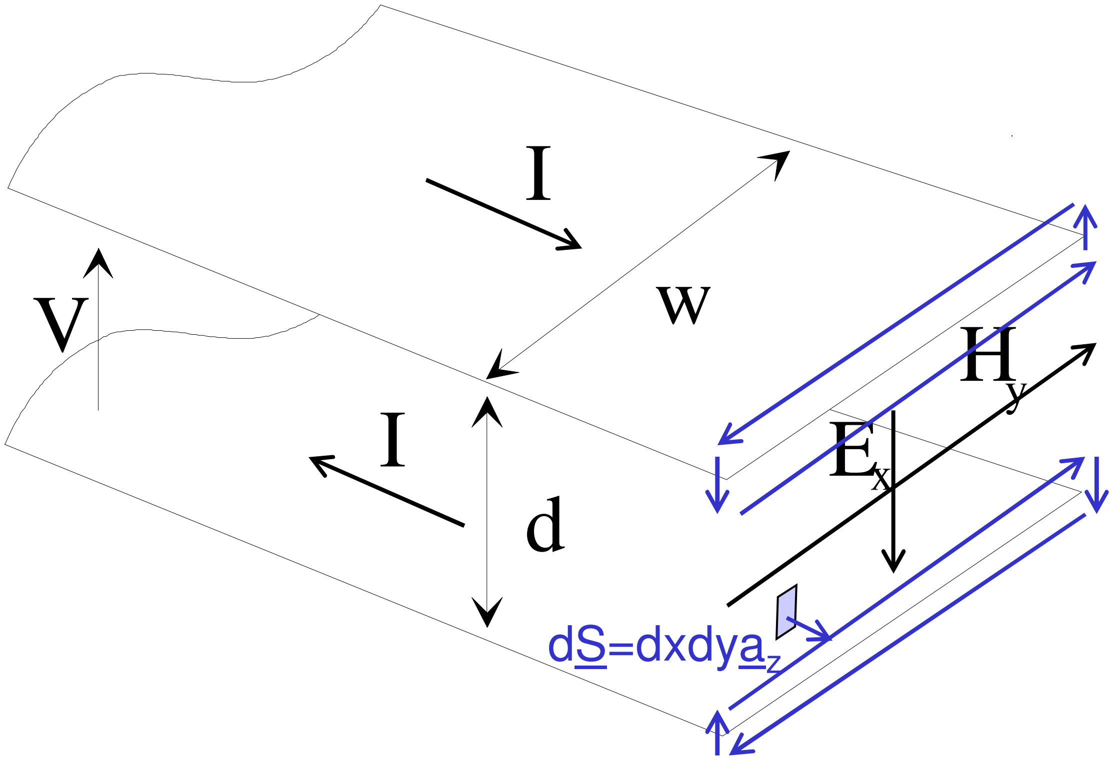
From the Maxwell-Ampere law:

$$\oint_c \underline{H}.d\underline{l} = \int_s \left(\underline{J} + \underline{\dot{D}}\right).d\underline{S}$$

But $\underline{\dot{D}}$ is in the same direction as E_x. Therefore it is orthogonal to $d\underline{S} = dxdy\underline{a}_z$ Thus:

$$\int_s \underline{\dot{D}} \cdot d\underline{S} = 0$$

$$\Rightarrow \oint_c \underline{H}.d\underline{l} = \int_s \underline{J}.d\underline{S}$$



We consider W>>d and neglect fringe effects

Solving for the top plate:

$$2H_y W = I$$

Solving for the bottom plate

$$2H_y W = I$$

Since the current is I on the top plate and –I on the bottom, the H fields sum inside the transmission line, while they cancel outside:

$$H_y^{Inside} = \frac{I}{2W} + \frac{I}{2W} = \frac{I}{W}$$

$$H_y^{Outside} = \frac{I}{2W} - \frac{I}{2W} = 0$$

$$I = H_y W$$

The electric field and voltage are related by:

$$E_x d = V$$

The transmission line average power is:

$$P_{Av} = \frac{1}{2}\mathrm{Re}\left\{\overline{V}\,\overline{I}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\overline{E}_x \overline{H_y}^*\right\}Wd$$

Wd=Area of the line

The average power density of the electromagnetic wave is then

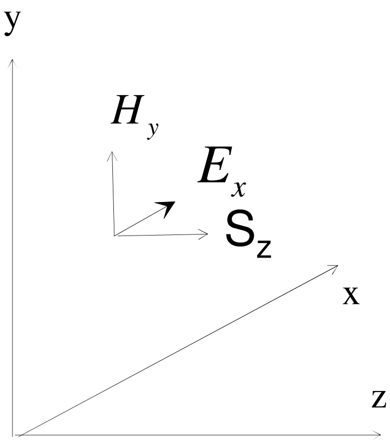
$$\frac{P_{Av}}{Area} = \frac{1}{2}\mathrm{Re}\left\{\overline{E}_x \overline{H_y}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\underline{\overline{E}} \times \underline{\overline{H}}^*\right\} \quad \left[\frac{W}{m^2}\right]$$

$$\underline{\overline{S}} = \frac{1}{2}\underline{\overline{E}} \times \underline{\overline{H}}^*$$ Complex Poynting Vector J. H. Poynting 1884

This gives the direction of power flow, which is is perpendicular to both $\underline{E}$ *and* $\underline{H}$

For a plane wave $\underline{E} = \left(E_x, 0, 0\right)$ and $\underline{H} = \left(0, H_y, 0\right)$

Thus $\underline{\overline{S}} = \frac{1}{2} E_x H^*_y \underline{a_z}$



Since $\eta = \frac{E_x}{H_y}$

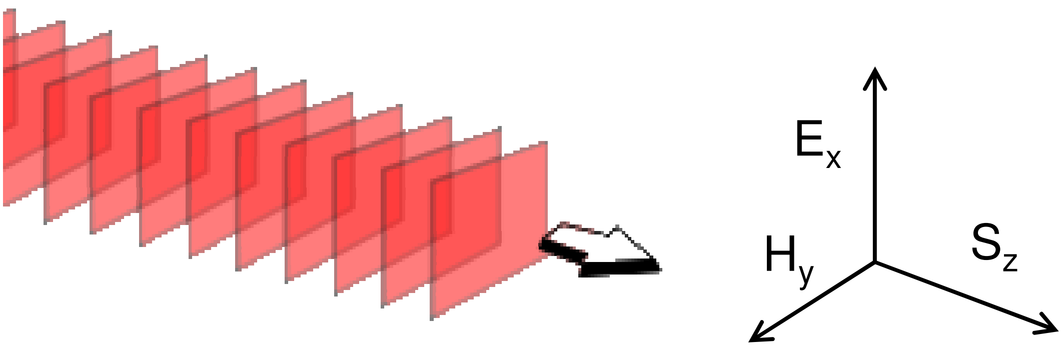
$$\text{Average Power} = \frac{\left|E_x\right|^2}{2\eta}$$

C.f. in a transmission line= $\frac{\left|\overline{V_F}\right|^2}{2Z_0}$

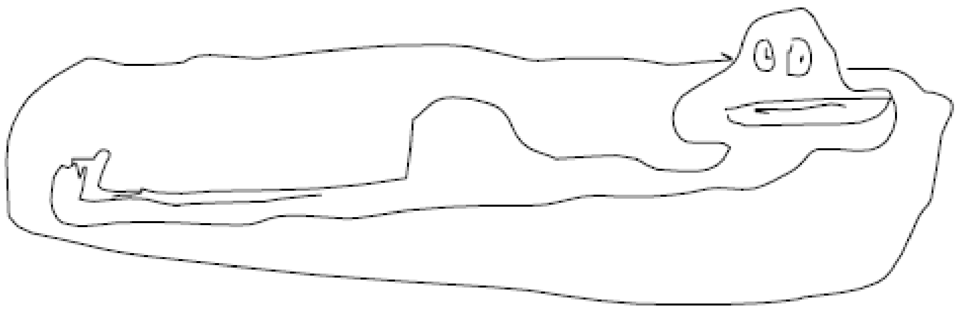
$\underline{S} = \underline{E} \times \underline{H}$ Poynting vector points in the direction of propagation

(note different definition with respect to Complex Poynting Vector)

Wavefront is locus of points having the same phase



II.2.6.1 Example – Duck a la microwave



A duck with a cross-sectional area of 0.1m^2 is heated in a microwave oven. If the electromagnetic wave is:

$$E_x = \mathrm{Re}\left\{750 e^{j(\omega t - \beta z)}\right\} Vm^{-1}, H_y = \mathrm{Re}\left\{2 e^{j(\omega t - \beta z)}\right\} Am^{-1}$$

What power is delivered to the duck ?

$$P = P_{Av} \cdot Area = \frac{1}{2}\mathrm{Re}\left(\underline{E} \times \underline{H}^*\right) \cdot Area = \frac{1}{2}(750 \cdot 2) \cdot 0.1 \approx 75W$$