P5-Electromagnetic Fields and Waves

Prof. Andrea C. Ferrari

http://www-g.eng.cam.ac.uk/nms/lecturenotes.html

CAPE Centre for Advanced Photonics and Electronics
www-cape.eng.cam.ac.uk

6 Lectures

3 Main Sections

~2 lectures per subject

I Transmission Lines

I.0 The wave equation

I.1 Telegrapher's Equations

I.2 Characteristic Impedance

I.3 Reflection

II Electromagnetic Waves in Free Space

III Antennae and Radio Transmission

OBJECTIVES

As the frequency of electronic circuits rises, one can no longer assume that voltages and currents are instantly transmitted by a wire.

The objectives of this course are:

- Appreciate when a wave theory is needed
- Derive and solve simple transmission line problems
- Understand the importance of matching to the characteristic impedance of a transmission cable
- Understand basic principles of EM wave propagation in free space, across interfaces and the use of antennae

This course deals with transmission of electromagnetic waves

1) along a cable (i.e. a transmission line)
2) through free space (the ‘ether’).

In the first half of these lectures, we will derive the differential equations which describe the propagation of a wave along a transmission line.

Then we will use these equations to demonstrate that these waves exhibit reflection, have impedance, and transmit power.

In the second half of these lectures we will look at the behaviour of waves in free space.

We will also consider different types of antennae for transmission and reception of electromagnetic waves.

Reference: OLVER A.D.
Microwave and Optical Transmission
John Wiley & Sons, 1992, 1997
Shelf Mark: NV 135

Handouts

The handouts have some gaps for you to fill

NOTE:

1) DO NOT PANIC IF YOU DO NOT MANAGE TO WRITE DOWN IN “REAL TIME”

2) Prefer to just sit back and relax?

You will be able to Download a PDF of the complete slides from

http://www-g.eng.cam.ac.uk/nms/lecturenotes.html

I.0 The Wave Equation

Aims

To recall basic phasors concepts
To introduce the generalised form of the wave equation

Objectives

At the end of this section you should be able to recognise the generalized form of the wave equation, its general solution, the propagation direction and velocity

I.0.0 Introduction

An ideal transmission line is defined as:
"a link between two points in which the signal at any point equals the initiating signal"

i.e. transmission takes place instantaneously and there is no attenuation

Real world transmission lines are not ideal, there is attenuation and there are delays in transmission

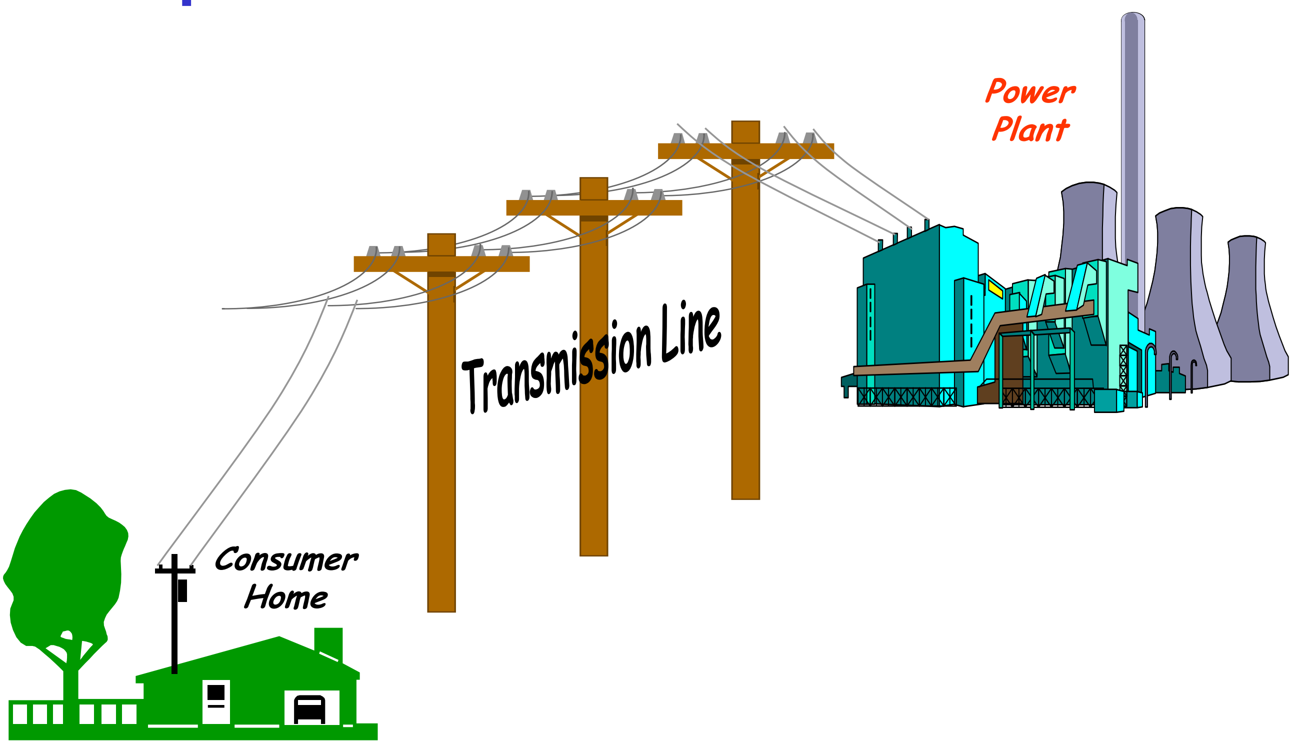
A transmission line can be seen as a device for propagating energy from one point to another

The propagation of energy is for one of two general reasons:

1. <u>Power transfer</u> (e.g. for lighting, heating, performing work) - examples are mains electricity, microwave guides in a microwave oven, a fibre-optic illuminator.

2. <u>Information transfer</u>. examples are telephone, radio, and fibre-optic links (in each case the energy propagating down the transmission line is modulated in some way).

Examples



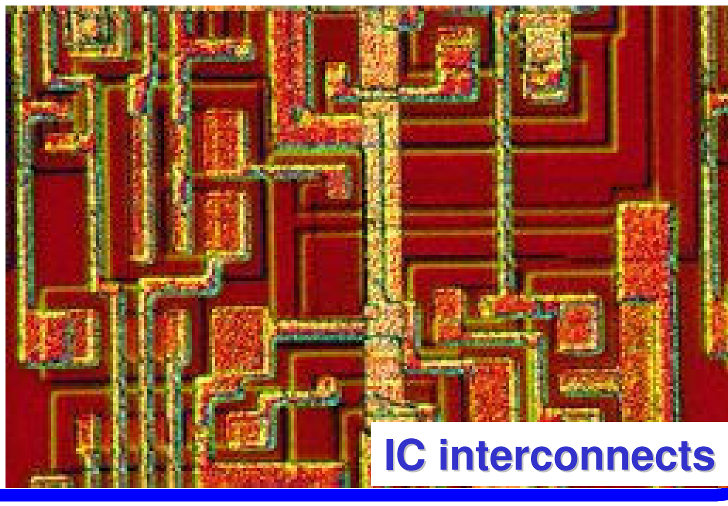
Antenna

Optical Fibre Link

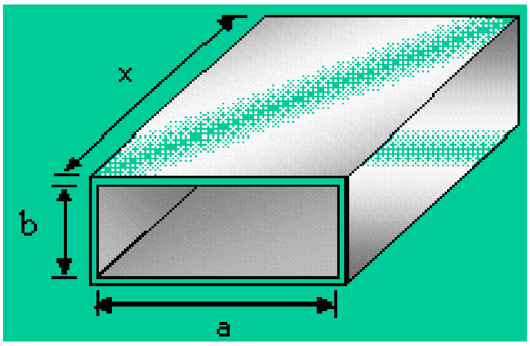
Pair of wires

Co-ax cable

PCB tracks

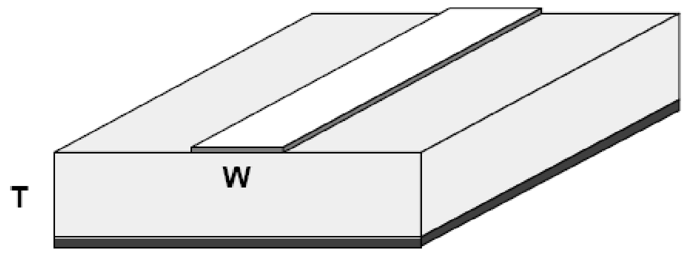


IC interconnects



Waveguides

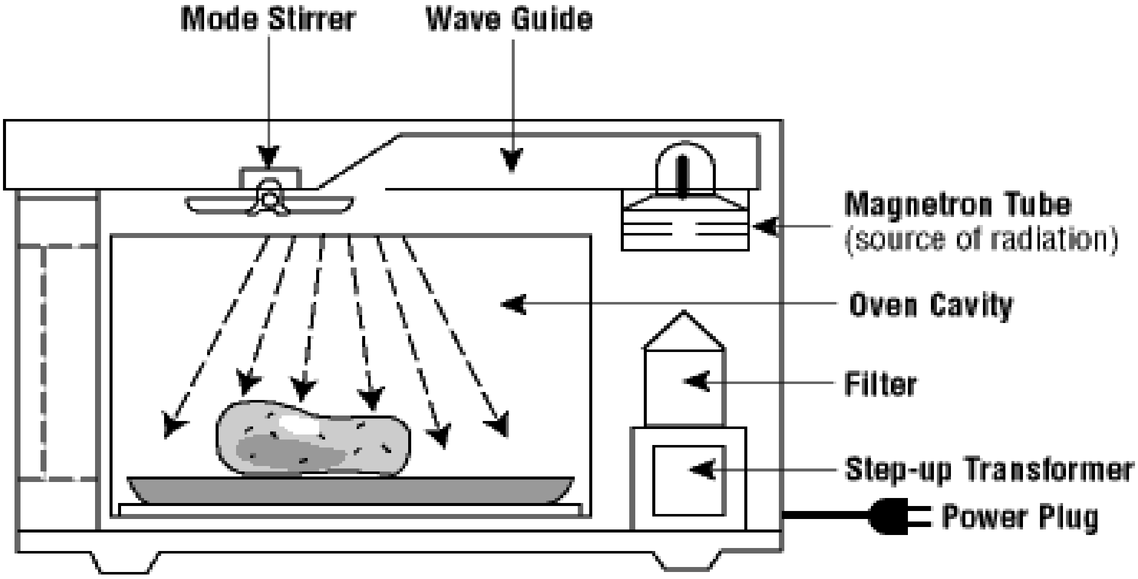
Mircostrip



Microstrip

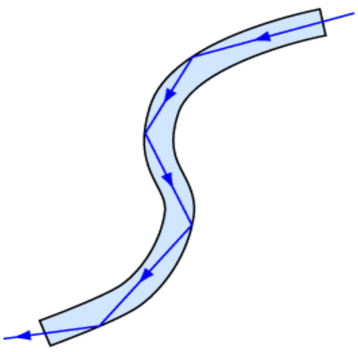
Dielectric of thickness T, with a conductor deposited on the bottom surface, and a strip of conductor of width W on the top surface

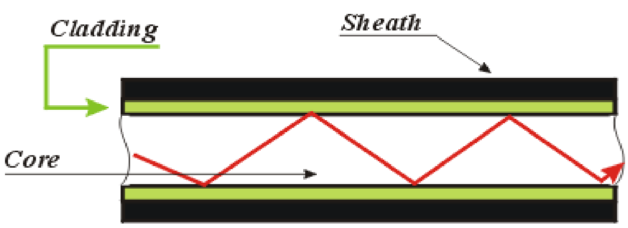
Can be fabricated using Printed Circuit Board (PCB) technology, and is used to convey microwave frequency signals

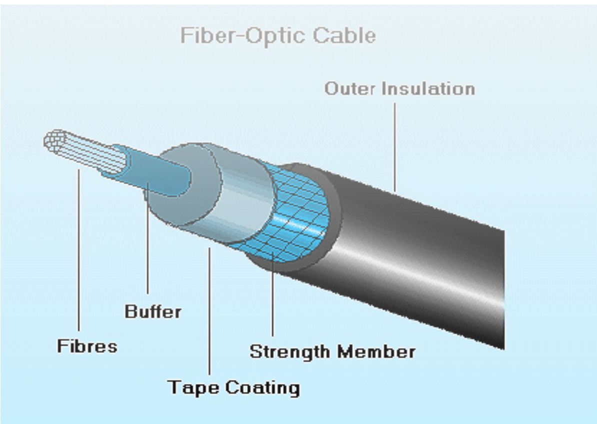


Microwave Oven

Optical Fibres



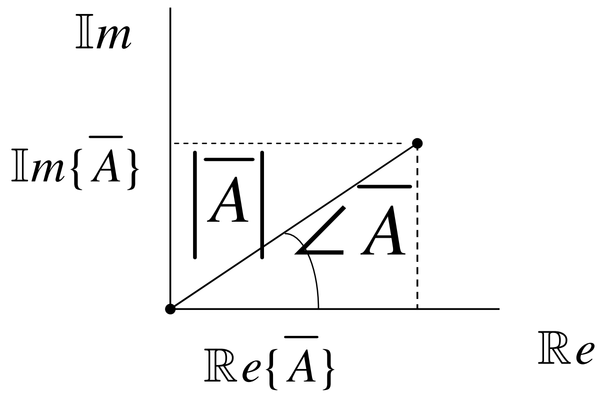




Phasor Notation

$\overline{A}$ means A is complex

$$\overline{A} = \mathbb{R}e\{\overline{A}\} + \mathbb{I}m\{\overline{A}\}\,j = \left|\overline{A}\right| e^{j\angle\overline{A}}$$



$$\left|\overline{A}\right| = A$$

$\overline{A}e^{j\beta x}$ is short-hand for $\mathbb{R}e\left\{\overline{A}e^{j(\beta x+\omega t)}\right\}$

which equals: $A\cos\left(\beta x + \omega t + \angle\overline{A}\right)$

Proof

$$e^{j(\pm\theta)} = \cos(\theta) \pm j\sin(\theta)$$

then

$$\overline{A}e^{j(\beta x+\omega t)} = Ae^{j\angle\overline{A}}e^{j(\beta x+\omega t)} = Ae^{j(\beta x+\omega t+\angle\overline{A})}$$

$$Ae^{j\left(\beta x+\omega t+\angle\overline{A}\right)} = A\cos(\beta x + \omega t + \angle\overline{A}) + jA\sin(\beta x + \omega t + \angle\overline{A})$$

I.0.1 The Wave Equation

The generalised form of the wave equation is:

$$\frac{\partial^2 A}{\partial t^2} = v^2 \nabla^2 A$$

Where the Laplacian of a scalar A is:

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$$

We will be looking at plane waves for which the wave equation is one-dimensional and appears as follows:

$$\frac{\partial^2 A}{\partial t^2} = v^2 \frac{\partial^2 A}{\partial x^2}$$

or

$$\frac{\partial^2 A}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 A}{\partial t^2}$$

Where A could be:

Either the Voltage (V) or the Current (I)
as in waves in a transmission line

Or the Electric Field (E) or Magnetic Field (H)
as in electromagnetic waves in free space

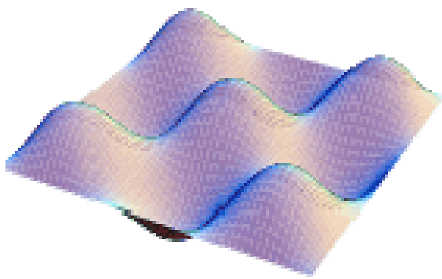
There are many other cases where the wave equation is used

For example

1) Waves on a string. These are planar waves where A represents the amplitude of the wave

2) Waves in a membrane, where there is variation in both x and y, and the equation is of the form

$$\frac{\partial^2 A}{\partial t^2} = v^2\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right)$$



The constant v is called the wave speed.

This comes from the fact that the general solution to the wave equation (D’Alembert solution,~1747) is

$$A = f(x \pm vt)$$

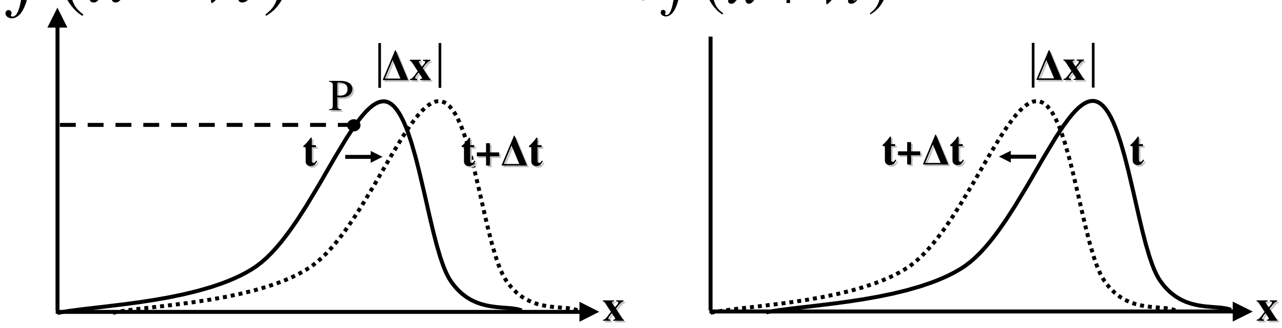
Note

$$A = f(x - vt)$$ **Forward moving**

$$A = f(x + vt)$$ **Backward moving**

Direction of travel

$f(x-vt)$ $\qquad\qquad$ $f(x+vt)$



Consider a fixed point, P, on the moving waveform, i.e. a point with constant *f*

f(x-vt) will be constant if *x-vt* is constant

If t increases ($t \rightarrow t+\Delta t$), *x* must also increase if *x-vt* is to be constant

An x increase implies that the wave is moving to the right (Forward)

Similarly, for x+vt ⇒ wave is moving to left (Backward)

Verify that $A = f(x \pm vt)$ is general solution

$$\frac{\partial A}{\partial t} = \pm v f'(x \pm vt) \qquad \frac{\partial^2 A}{\partial t^2} = v^2 f''(x \pm vt)$$

$$\frac{\partial A}{\partial x} = f'(x \pm vt) \qquad \frac{\partial^2 A}{\partial x^2} = f''(x \pm vt)$$

$$\frac{\partial^2 A}{\partial t^2} = v^2 \frac{\partial^2 A}{\partial x^2}$$

I.1 Electrical Waves

Aims

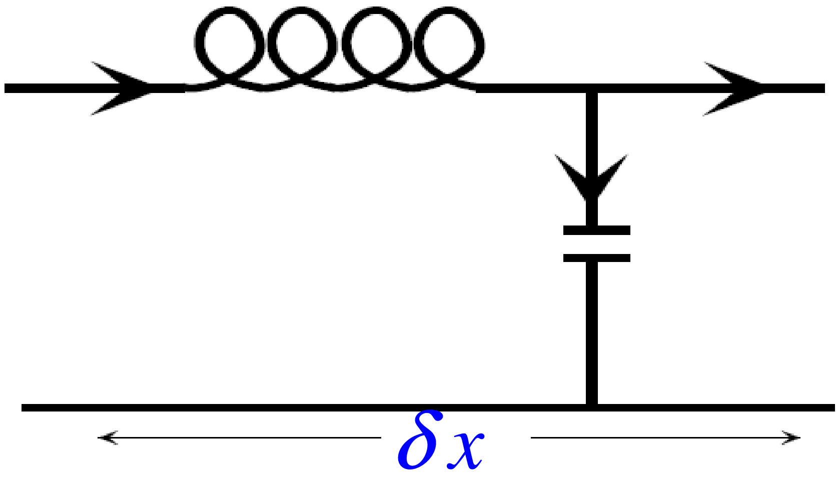
To derive the telegrapher's equations
To account for losses in transmission lines

Objectives

At the end of this section you should be able to recognise when the wave theory is relevant; to master the concepts of wavelenght, wave velocity, period and phase; to describe the propagation of waves in loss-less and lossy transmission lines

I.1.1 Telegrapher's Equations

Let us consider a short length, δx, of a wire pair



This could, for example, represent a coaxial cable

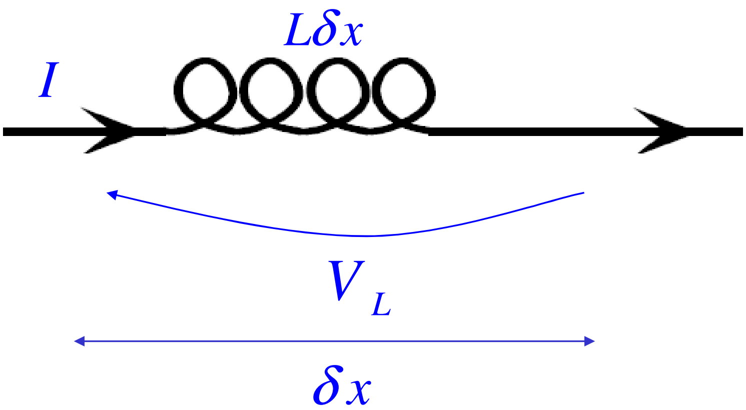
For a small δx, any function A(x) can be written as

$$A(x+\delta x) \approx A(x) + \frac{\partial A(x)}{\partial x}\delta x$$

In our case A can be Voltage (V) or Current (I)

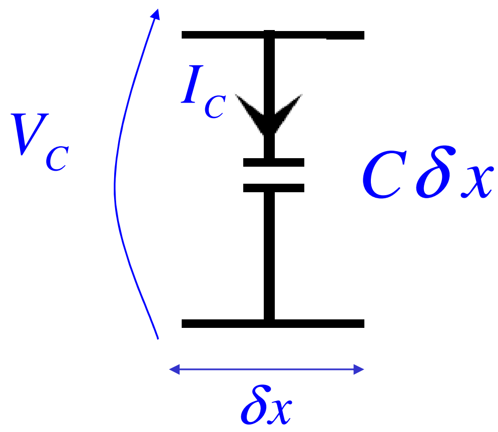
Let us define

L series/loop inductance per unit length [H/m]

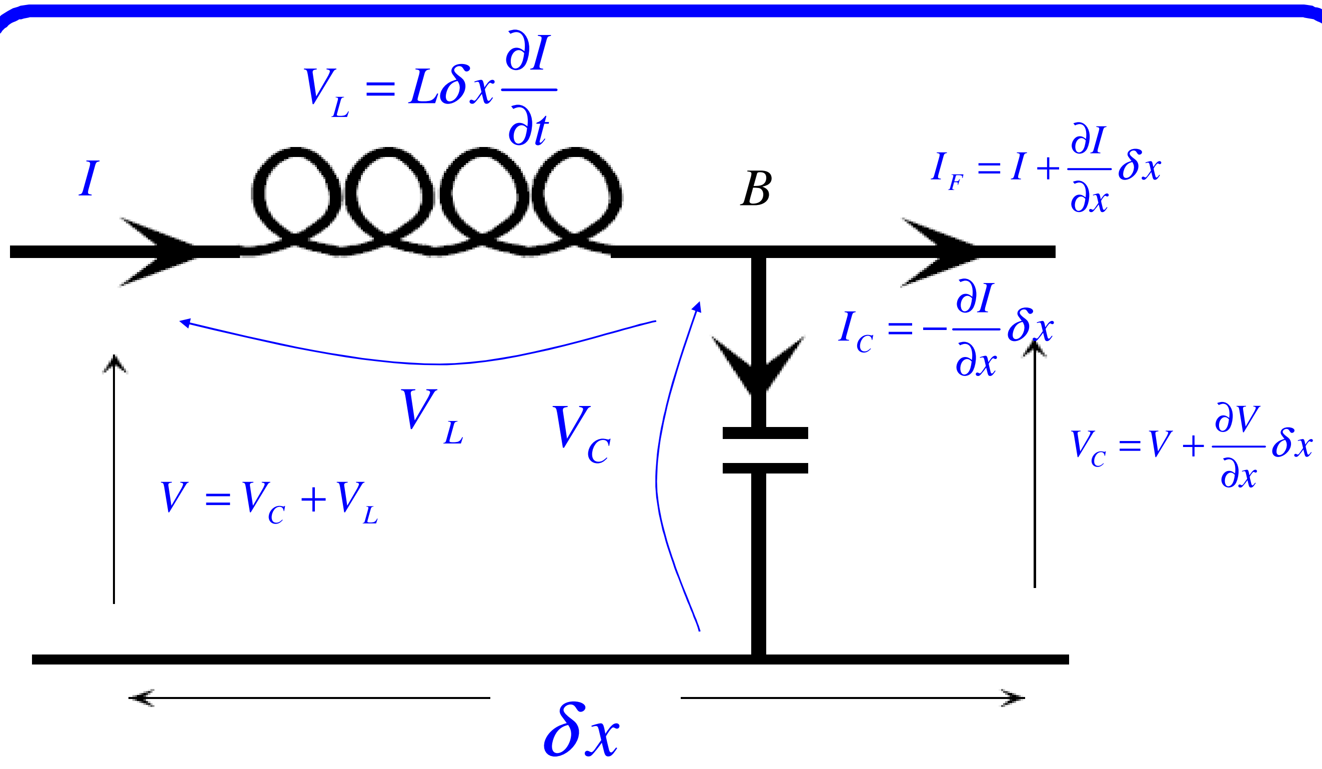


$$V_L = L\delta x \frac{\partial I}{\partial t}$$

C parallel/shunt capacitance per unit length [F/m]



$$I_C = C\delta x \frac{\partial V_C}{\partial t}$$



$$I_C = C\delta x\frac{\partial V_C}{\partial t} = C\delta x\frac{\partial}{\partial t}(V + \frac{\partial V}{\partial x}\delta x) = C\delta x\frac{\partial V}{\partial t} + C\frac{\partial V}{\partial x\partial t}(\delta x)^2 \approx C\delta x\frac{\partial V}{\partial t}$$

$$V_C = V - V_L \quad \Rightarrow \quad \cancel{V} + \frac{\partial V}{\partial x}\cancel{\delta x} = \cancel{V} - L\frac{\partial I}{\partial t}\cancel{\delta x}$$

$$I_F = I - I_C \quad \Rightarrow \quad \cancel{I} + \frac{\partial I}{\partial x}\cancel{\delta x} = \cancel{I} - \frac{\partial V}{\partial t}C\cancel{\delta x}$$

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \qquad \textbf{(1.1)}$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \qquad \textbf{(1.2)}$$

Eqs. (1.1),(1.2) are known as the **"telegrapher's equations"**
They were derived in 1885 by Oliver Heaviside, and were crucial in the early development of long distance telegraphy (hence the name)

I.1.2 Travelling Wave Equations

Let us differentiate both (1.1) and (1.2) with respect to x

$$\frac{\partial^2 V}{\partial x^2} = -L\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial x}\right) \qquad = LC\frac{\partial^2 V}{\partial t^2} \qquad \text{(1.1a)}$$

$$\frac{\partial^2 I}{\partial x^2} = -C\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right) \qquad = LC\frac{\partial^2 I}{\partial t^2} \qquad \text{(1.2a)}$$

Then in (1.1a) substitute $\frac{\partial I}{\partial x}$ using (1.2)

Then in (1.2a) substitute $\frac{\partial V}{\partial x}$ using (1.1)

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \qquad \text{(1.1a)}$$

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} \qquad \text{(1.2a)}$$

Same functional form as wave equation:

$$\frac{\partial^2 A}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 A}{\partial t^2} \quad \Rightarrow \quad v^2 = \frac{1}{LC}$$

We try a solution for V in (1.1a) of the form

$$\overline{V} = \overline{A}e^{j\beta x}e^{j\omega t}$$

$$\cancel{-}\beta^2 \cancel{\overline{A}}\cancel{e^{j\beta x}}\cancel{e^{j\omega t}} = \cancel{-}\omega^2 LC \cancel{\overline{A}}\cancel{e^{j\beta x}}\cancel{e^{j\omega t}}$$

Hence

$$\beta = \pm\omega\sqrt{LC}$$ **Phase Constant (1.3)**

Since β can be positive or negative, we obtain expressions for voltage and current waves moving forward (subscript F) and backward (subscript B) along the transmission line

$$V = \mathbb{R}e\left\{\left(\overline{V_F}e^{-j\beta x} + \overline{V_B}e^{j\beta x}\right)e^{j\omega t}\right\} \qquad \textbf{(1.4)}$$

$$I = \mathbb{R}e\left\{\left(\overline{I_F}e^{-j\beta x} + \overline{I_B}e^{j\beta x}\right)e^{j\omega t}\right\} \qquad \textbf{(1.5)}$$

I.1.3 Lossy Transmission Lines

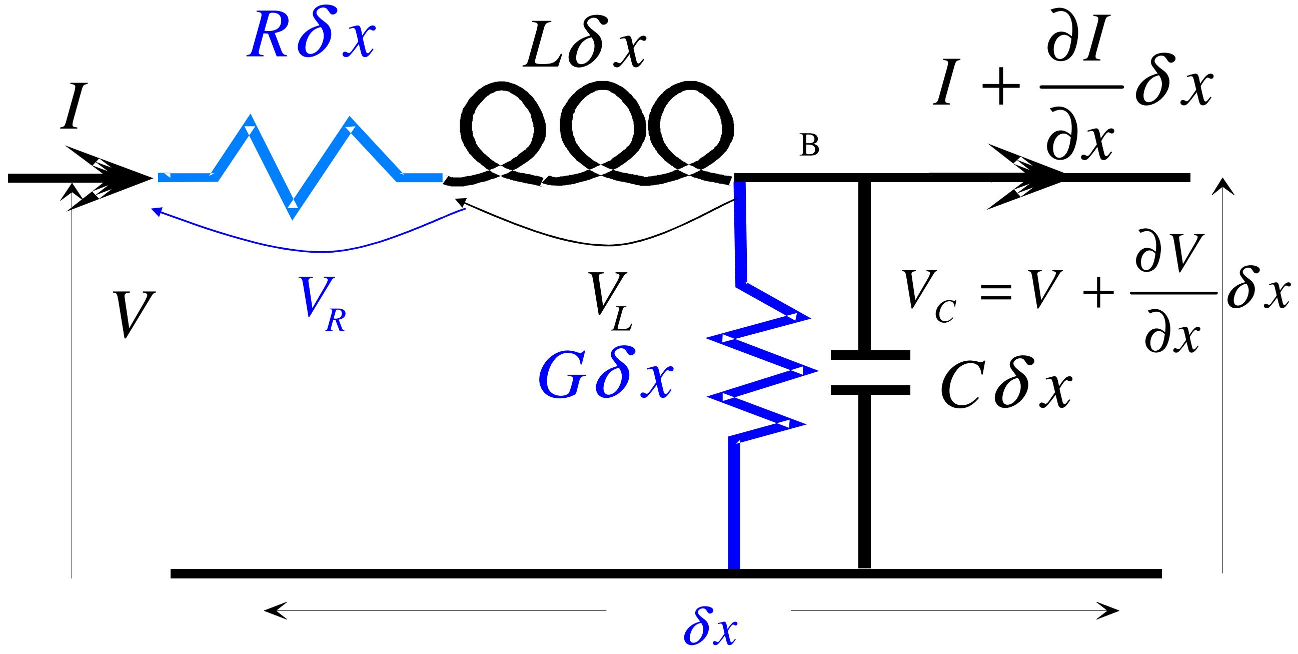
Thus far we considered a lossless transmission line. Therefore we did not include any resistance along the line, nor any conductance across the line.

If we now define

R= series resistance per unit length [Ω/m]

G= shunt conductance per unit length [S/m]

To derive the relevant expressions for a lossy transmission line our equivalent circuit would become :



$$V - V_R - V_L - V_C = 0$$

$$V - R\delta x I - L\delta x\frac{\partial I}{\partial t} - \left(V + \frac{\partial V}{\partial x}\delta x\right) = 0$$

$$\frac{\partial V}{\partial x} = -\left(RI + L\frac{\partial I}{\partial t}\right)$$

For simplicity we assume

$$I = f(x)e^{j\omega t} \quad \Rightarrow \quad \frac{\partial I}{\partial t} = j\omega f(x)e^{j\omega t} = j\omega I$$

Then

$$\frac{\partial V}{\partial x} = -(R + j\omega L)I$$

Compare with (1.1) $\quad \frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \quad = -j\omega LI$

Similarly, using Kirchoff's current law to sum currents:

$$\cancel{I} - G\cancel{\delta} xV - j\omega C\cancel{\delta} xV - (\cancel{I} + \frac{\partial I}{\partial x}\cancel{\delta} x) = 0$$

$$\frac{\partial I}{\partial x} = -(G + j\omega C)V$$

Compare with (1.2) $$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} = -j\omega CV$$

Thus, we can write the expression for a lossy line starting from that of a lossless line, if we substitute

L in a lossless line with:

$$L' = \frac{(R + j\omega L)}{j\omega}$$ in a lossy line

C in a lossless line with:

$$C' = \frac{(G + j\omega C)}{j\omega}$$ in a lossy line

Then $\beta = \omega\sqrt{LC}$ In a lossless line corresponds to:

$$\beta' = \frac{1}{j}\sqrt{(R + j\omega L)(G + j\omega C)}$$

in a lossy line

Substituting β ⇒ β' into (1.4) and (1.5) and defining

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

We get

$$V = \mathbb{R}e\left\{\left(\overline{V_F} e^{-(\alpha + j\beta)x} + \overline{V_B} e^{(\alpha + j\beta)x}\right) e^{j\omega t}\right\} \qquad (1.6)$$

$$I = \mathbb{R}e\left\{\left(\overline{I_F} e^{-(\alpha + j\beta)x} + \overline{I_B} e^{(\alpha + j\beta)x}\right) e^{j\omega t}\right\} \qquad (1.7)$$

γ is called propagation constant

β is the phase constant

The real term α corresponds to the attenuation along the line and is known as the **attenuation constant**

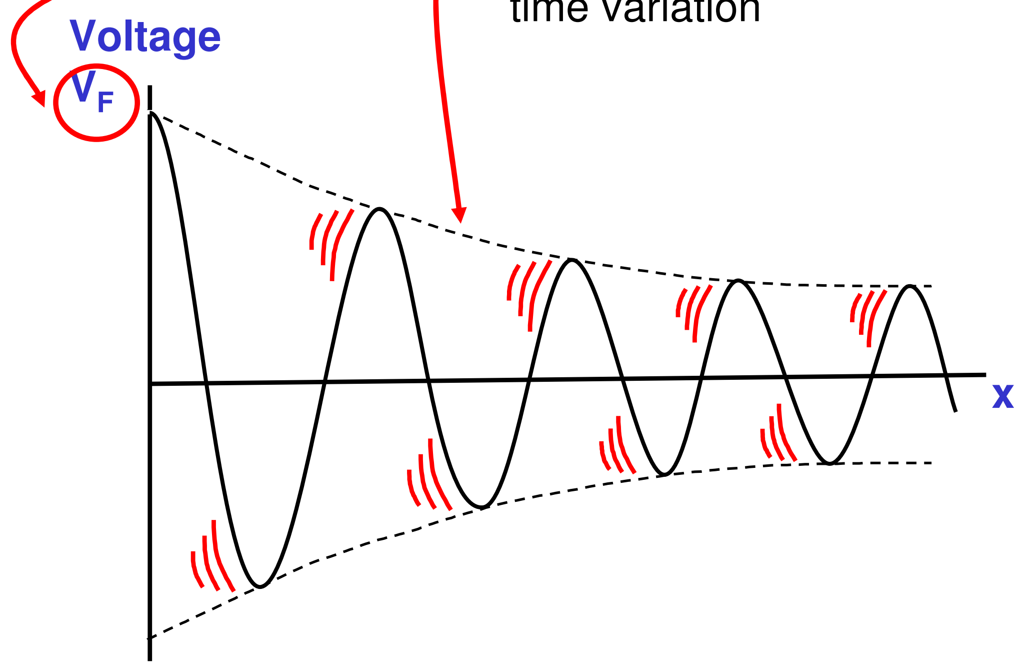
For a forward travelling wave:

$V = V_F\, e^{j\omega t}\, e^{-\gamma x} = V_F\, e^{-\alpha x}\, e^{j(\omega t - \beta x)}$ $\Rightarrow$ $|V| = |V_F|\, e^{-\alpha x}$

amplitude factor

phase factor

time variation



Note:

At high frequencies: $\omega L >> R$ and $\omega C >> G$:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \approx \sqrt{-\omega^2 LC} = j\omega\sqrt{LC}$$

Thus

$$\alpha \approx 0 \qquad \beta \approx \omega\sqrt{LC}$$

The expressions approximate back to those for lossless lines

I.1.4 Wave velocity: v

Our expressions for voltage and current contain 2 exponentials

The one in terms of x: $e^{\pm j\beta x}$

gives the spatial dependence of the wave, hence the wavelength:

$$\lambda = \frac{2\pi}{\beta}$$

The other: $e^{j\omega t}$

gives the temporal dependence of the wave, hence its frequency:

$$f = \frac{\omega}{2\pi}$$

For a wave velocity v, wavelength λ and frequency f:

$$v = f\lambda$$

then

$$v = \frac{\omega}{2\pi}\frac{2\pi}{\beta}$$

since $\beta = \omega\sqrt{LC}$

$$\Rightarrow \quad v = \frac{1}{\sqrt{LC}} \qquad (1.8)$$

I.1.5 Example: Wavelength

An Ethernet cable has L= 0.22 μHm^{-1} and C = 86 pFm^{-1}.

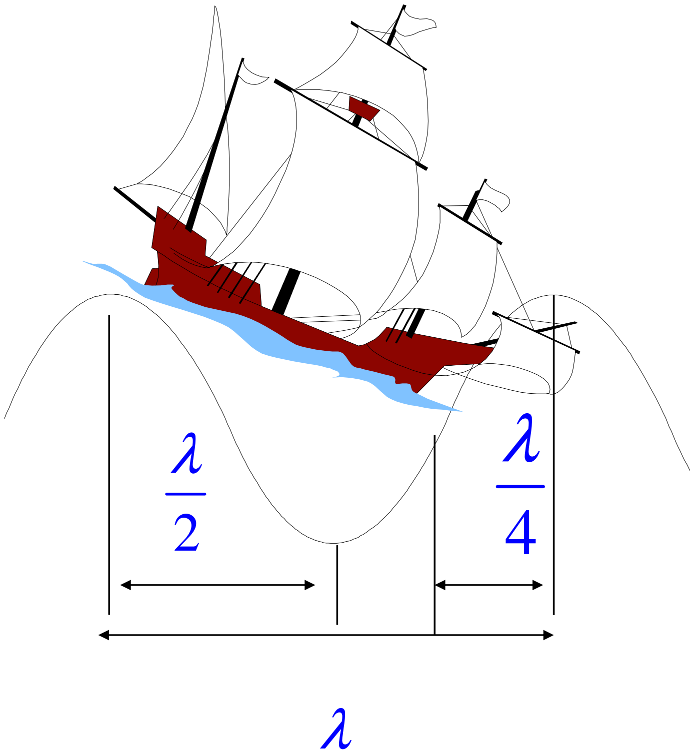
What is the wavelength at 10 MHz ?

From $\lambda = \frac{2\pi}{\beta}$ and $\beta = \omega\sqrt{LC}$

$$\Rightarrow \quad \lambda = \frac{2\pi}{\omega\sqrt{LC}}$$

Then $\lambda = \frac{2\pi}{2\pi \cdot 10 \cdot 10^{6}\sqrt{0.22 \cdot 10^{-6} \cdot 86 \cdot 10^{-12}}} = 23 metres$

I.1.6 When must distances be accounted for in AC circuits?



Large ship is in serious trouble (as you can see) and we cannot ignore the effect of the waves

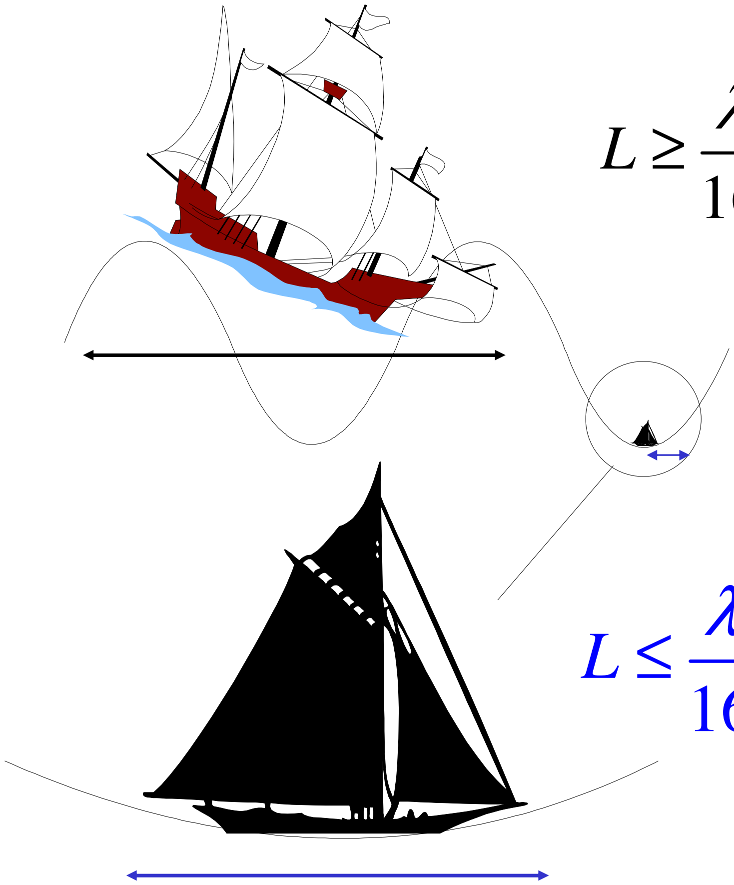
A much smaller vessel caught in the same storm fares much better

If a circuit is one quarter of a wavelength across, then one end is at zero, the other at a maximum

If a circuit is an eighth of a wavelength across, then the difference is $\sqrt{2}$ of the amplitude

In general, if the wavelength is long in comparison to our electrical circuit, then we can use standard circuit analysis without considering transmission line effects.

A good rule of thumb is for the wavelength to be a factor of 16 longer



$$L \geq \frac{\lambda}{16}$$ **Wave Relevant**

$$L \leq \frac{\lambda}{16}$$ **Wave Not Relevant**

I.1.7 Example: When is wave theory relevant?

A designer is creating a circuit which has a clock rate of 5MHz and has 200mm long tracks for which the inductance (L) and capacitance (C) per unit length are:

$L=0.5\mu Hm^{-1}$ $C=60pFm^{-1}$

From $\lambda = \dfrac{2\pi}{\beta}$ And $\beta = \omega\sqrt{LC}$ $\Rightarrow$ $\lambda = \dfrac{2\pi}{\omega\sqrt{LC}}$

Then $$\lambda = \frac{2\pi}{2\pi \cdot 5 \cdot 10^{6}\sqrt{0.5 \cdot 10^{-6} \cdot 60 \cdot 10^{-12}}} = 36.5m$$

36.5 m is much greater than 200 mm (the size of the circuit board), so that wave theory is irrelevant.

Note: The problem is even less relevant for mains frequencies i.e. 50 Hz. This gives λ~3650 km

I.2 Characteristic Impedance

Aims

To define and derive the characteristic impedance for lossless and lossy lines

Objectives

At the end of this section you should be able to describe the forward and backward waves in a transmission line and calculate the characteristic impedance

I.2.1 Lossless Lines

Recalling the solutions for I & V (equations 1.4&1.5):

$$V = \mathbb{R}e\left\{\left(\overline{V_F}e^{-j\beta x} + \overline{V_B}e^{j\beta x}\right)e^{j\omega t}\right\} \quad \textbf{(1.4)}$$

$$I = \mathbb{R}e\left\{\left(\overline{I_F}e^{-j\beta x} + \overline{I_B}e^{j\beta x}\right)e^{j\omega t}\right\} \quad \textbf{(1.5)}$$

Differentiating (1.5) with respect to x

(1.4) with respect to t and multiplying by -C

$$\frac{\partial I}{\partial x} = \mathbb{R}e\left\{\left(-j\beta\overline{I_F}e^{-j\beta x} + j\beta\overline{I_B}e^{j\beta x}\right)e^{j\omega t}\right\} \quad \textbf{(2.1)}$$

$$-C\frac{\partial V}{\partial t} = \mathbb{R}e\left\{\left(-Cj\omega\overline{V_F}e^{-j\beta x} - Cj\omega\overline{V_B}e^{j\beta x}\right)e^{j\omega t}\right\} \quad \textbf{(2.2)}$$

According to the second Telegrapher's equation:

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \qquad (1.2)$$

We can then equate (2.1) and (2.2):

$$\mathbb{R}e\left\{\left(-j\beta\overline{I_F}e^{-j\beta x} + j\beta\overline{I_B}e^{j\beta x}\right)e^{j\omega t}\right\} = \mathbb{R}e\left\{\left(-Cj\omega\overline{V_F}e^{-j\beta x} - Cj\omega\overline{V_B}e^{j\beta x}\right)e^{j\omega t}\right\}$$

Since $e^{j(\omega t-\beta x)}$ and $e^{j(\omega t+\beta x)}$

represent waves travelling in opposite directions, they can be treated separately.

This leads to two independent expressions in V and I

$$\cancel{-j}\beta\overline{I_F}\cancel{e^{-j\beta x}} = \cancel{-}C\cancel{j}\omega\overline{V_F}\cancel{e^{-j\beta x}}$$

$$\frac{\overline{V_F}}{\overline{I_F}} = \frac{\beta}{C\omega}$$

$$\overline{I_B}\cancel{j}\beta\cancel{e^{j\beta x}} = -C\cancel{j}\omega\overline{V_B}\cancel{e^{j\beta x}}$$

$$\frac{\overline{V_B}}{\overline{I_B}} = -\frac{\beta}{C\omega}$$

Note: If we consider $\overline{V_F}$ and $\overline{V_B}$ to have the same sign then, due to the differentiation with respect to x, $\overline{I_B}$ and $\overline{I_F}$ have opposite signs

The characteristic impedance, Z_0 is defined as the ratio between the voltage and the current of a **unidirectional forward** wave on a transmission line at any point, with no reflection:

$$Z_0 = \frac{\overline{V_F}}{\overline{I_F}}$$

Z_0 is always positive

Since $\frac{\overline{V_F}}{\overline{I_F}} = \frac{\beta}{C\omega}$ $\Rightarrow$ $Z_0 = \frac{\beta}{C\omega}$

From (1.3) $\beta = \omega\sqrt{LC}$

$$\Rightarrow \quad Z_0 = \sqrt{\frac{L}{C}} \qquad \textbf{(2.3)}$$

Units

$[\beta] = m^{-1}$ $\qquad [\omega] = s^{-1}$

$$[C] = \frac{F}{m} = \frac{1}{m}\frac{A \cdot s}{V} = \frac{s}{\Omega \cdot m} \qquad [L] = \frac{H}{m} = \frac{1}{m}\frac{V \cdot s}{A} = \frac{\Omega \cdot s}{m}$$

$$[Z_0] = \left[\frac{\overline{V}}{\overline{I}}\right] = \left[\sqrt{\frac{L}{C}}\right] = \Omega$$

Z_0 is the total impedance of a line of any length if there are no reflections $\Rightarrow$ I and V in phase everywhere. Z_0 is real

If there are reflections, the current and voltage of the advancing wave are again in phase, but not necessarily with the current and voltage of the retreating wave

The lossless line has no resistors. Yet Z_0 has units of Ω.

The characteristic impedance does not dissipate power. It stores it

I.2.2 Lossy Lines

$$V = \mathbb{R}e\left\{\left(\overline{V_F}e^{-(\alpha+j\beta)x} + \overline{V_B}e^{(\alpha+j\beta)x}\right)e^{j\omega t}\right\} \qquad (1.6)$$

$$I = \mathbb{R}e\left\{\left(\overline{I_F}e^{-(\alpha+j\beta)x} + \overline{I_B}e^{(\alpha+j\beta)x}\right)e^{j\omega t}\right\} \qquad (1.7)$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Remembering that we can write the expressions for a lossy line starting from those of a lossless line, if we substitute

L in a lossless line with:

$$L' = \frac{(R + j\omega L)}{j\omega}$$ in a lossy line

C in a lossless line with:

$$C' = \frac{(G + j\omega C)}{j\omega}$$ in a lossy line

Thus $Z_0 = \sqrt{\frac{L}{C}}$ in a lossless line

corresponds to $\overline{Z_0} = \sqrt{\frac{L'}{C'}}$ in a lossy line

$$\Rightarrow \quad \overline{Z_0} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Note: at high frequencies $\omega L >> R$ and $\omega C >> G$,

we recover the expression for lossless lines

$$Z_0 \approx \sqrt{\frac{L}{C}}$$

I.2.3 Summary

1) For a unidirectional wave:

$V = Z_0 I$ at all points

2) For any wave:

$\overline{V_F} = Z_0 \overline{I_F}$ and $\overline{V_B} = -Z_0 \overline{I_B}$

Hence $\overline{V_F}$ and $\overline{I_F}$ are in phase

$\overline{V_B}$ and $\overline{I_B}$ are in antiphase

3) For a lossless line Z_0 is real with units of ohms.

4) For a lossy line $\overline{Z_0}$ is complex

I.2.4 Characteristic Impedance – Example 1

Q: We wish to examine a circuit using an oscilloscope. The oscilloscope probe is on an infinitely long cable and has a characteristic impedance of 50 Ω.

What load does the probe add to the circuit?

A:

1) Since the cable is infinitely long there are no reflections

2) For a unidirectional wave with no reflections $Z_0 = V/I$ at all points, hence the probe behaves like a load of 50Ω

I.2.5 Characteristic Impedance – Example 2

Q: A wave of $\overline{V_F}$ = 5 volts with a wavelength λ=2 metres has a reflected wave of $\overline{V_B}$ = 1 volt

If Z_0 = 75Ω, what are the voltage and current 3 metres from the end of the cable?

A: From Equation 1.4:

$$\overline{V} = \overline{V_F} e^{-j\beta x} + \overline{V_B} e^{j\beta x}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{2m} = \pi \quad [m^{-1}]$$

x = - 3m therefore:

$$\overline{V} = 5e^{+j3\pi} + 1e^{-j3\pi} \quad \text{[volts]}$$

Since

$$\overline{I_F} = \frac{\overline{V_F}}{Z_0}$$

and

$$\overline{I_B} = -\frac{\overline{V_B}}{Z_0}$$

Then

$$\overline{I} = \frac{5}{75} e^{+j3\pi} - \frac{1}{75} e^{-j3\pi} \quad \text{[amps]}$$

I.3 Reflection

Aims

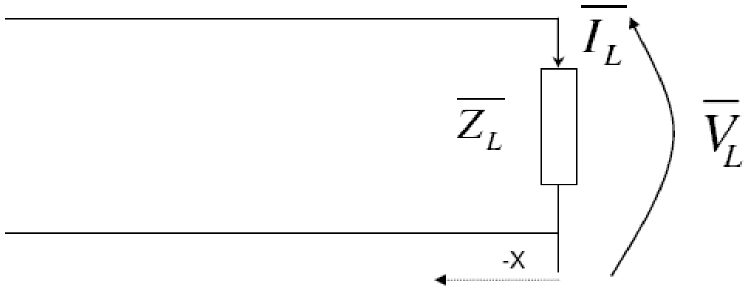
To introduce the concept of voltage reflection coefficient and its relation to the reflected power at the load

Objectives

At the end of this section you should be able to calculate the voltage reflection coefficient, the incident and reflected power on the load, the conditions for ringing and quarter wave matching

I.3.1 Voltage reflection coefficient

Consider a load added to the end of a transmission line



From Equation 1.4: $\overline{V} = \overline{V_F} e^{-j\beta x} + \overline{V_B} e^{j\beta x}$

From Equation 1.5: $\overline{I} = \overline{I_F} e^{-j\beta x} + \overline{I_B} e^{j\beta x}$

At the load x=0, thus

$$\overline{V} = \overline{V_F} + \overline{V_B} = \overline{V_L}$$

$$\overline{I} = \overline{I_F} + \overline{I_B} = \overline{I_L}$$

But: $\overline{V} = \overline{V_F} + \overline{V_B} = \overline{V_L} = \overline{Z_L}\,\overline{I_L} \quad = \overline{Z_L}\left(\overline{I_F} + \overline{I_B}\right)$

From our derivation of characteristic impedance:

$$\overline{I_F} = \frac{\overline{V_F}}{Z_0} \qquad \overline{I_B} = -\frac{\overline{V_B}}{Z_0}$$

$\overline{I_F}$ and $\overline{I_B}$ have opposite signs relative to $\overline{V_F}$ and $\overline{V_B}$

Hence: $\overline{V_F} + \overline{V_B} = \overline{Z_L}\left(\overline{I_F} + \overline{I_B}\right) = \overline{Z_L}\,\dfrac{\overline{V_F} - \overline{V_B}}{Z_0}$

$$\frac{\overline{V_B}}{\overline{V_F}} = \frac{\overline{Z_L} - Z_0}{\overline{Z_L} + Z_0}$$

The Voltage Reflection Coefficient, $\overline{\rho_L}$, is defined as the complex amplitude of the reverse voltage wave divided by the complex amplitude of the forward voltage wave at the load:

$$\overline{\rho_L} = \frac{\overline{V_B}}{\overline{V_F}} \qquad (3.1a)$$

$$\overline{\rho_L} = \frac{\overline{Z_L} - Z_0}{\overline{Z_L} + Z_0} \qquad (3.1b)$$

I.3.2 Power Reflection

At the load

$$V(t) = \mathrm{Re}\left\{\overline{V} e^{j\omega t}\right\} = V\cos(\omega t + \angle\overline{V})$$

$$I(t) = \mathrm{Re}\left\{\overline{I} e^{j\omega t}\right\} = I\cos(\omega t + \angle\overline{I})$$

Instantaneous power dissipated at the load:

$$P(t) = V(t)I(t) = VI\cos(\omega t + \angle\overline{V})\cos(\omega t + \angle\overline{I})$$

Remembering the identity:

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

we get:

$$P(t) = \frac{1}{2}VI\left[\cos(2\omega t + \angle\overline{V} + \angle\overline{I}) + \cos(\angle\overline{V} - \angle\overline{I})\right]$$

Mean power dissipated at the load:

$$P_{Av} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{2}VI\cos(\angle\overline{V} - \angle\overline{I}) = \frac{1}{2}\mathrm{Re}\left\{\overline{V}\,\overline{I}^*\right\}$$

Where $\overline{I}^*$ is the complex conjugate of $\overline{I}$

Thus $\overline{I} = \mathbb{R}e\{\overline{I}\} + \mathbb{I}m\{\overline{I}\}j = \left|\overline{I}\right|e^{j\angle\overline{I}}$ $\Rightarrow$ $\overline{I}^* = \mathbb{R}e\{\overline{I}\} - \mathbb{I}m\{\overline{I}\}j = \left|\overline{I}\right|e^{-j\angle\overline{I}}$

At the load:

$$\overline{V} = \overline{V_F} + \overline{V_B}$$

But, from (3.1a):

$$\overline{V_B} = \overline{\rho_L}\,\overline{V_F}$$

$$\Rightarrow \quad \overline{V} = \overline{V_F}(1 + \overline{\rho_L})$$

Similarly:

At the load:

$$\overline{I} = \overline{I}_F + \overline{I}_B = \frac{1}{Z_0}(\overline{V}_F - \overline{V}_B) = \frac{\overline{V}_F}{Z_0}\left(1 - \frac{\overline{V}_B}{\overline{V}_F}\right)$$

$$\Rightarrow \quad \overline{I} = \frac{\overline{V_F}}{Z_0}(1 - \overline{\rho_L})$$

Hence:

$$\frac{1}{2}\overline{V}\,\overline{I}^* = \frac{1}{2}\left(1 + \overline{\rho_L}\right)\left(1 - \overline{\rho_L}^*\right)\frac{\left|\overline{V_F}\right|^2}{Z_0}$$

$$= \frac{\left|\overline{V_F}\right|^2}{2Z_0}\left(1 + \overline{\rho_L} - \overline{\rho_L}^* - \left|\overline{\rho_L}\right|^2\right)$$

But $\overline{\rho}_L^*$ is the complex conjugate of $\overline{\rho}_L$

$\Rightarrow$ $\overline{\rho}_L - \overline{\rho}_L^*$ is imaginary

so: $\frac{1}{2}\mathrm{Re}\left\{\overline{V}\,\overline{I}^*\right\} = \frac{\left|\overline{V}_F\right|^2}{2Z_0}\left(1 - \left|\overline{\rho}_L\right|^2\right)$ power dissipated in the load

Therefore:

Incident power= $\frac{\left|\overline{V}_F\right|^2}{2Z_0}$ Reflected power= $\left|\overline{\rho}_L\right|^2 \frac{\left|\overline{V}_F\right|^2}{2Z_0}$

The fraction of power reflected from the load is:

$$\left|\overline{\rho}_L\right|^2$$

I.3.3 Standing Waves

Reflections result in standing waves being set up in the transmission line. The Voltage Standing Wave Ratio (VSWR) is a measurement of the ratio of the maximum voltage to the minimum voltage.

$$VSWR = \frac{\text{Maximum voltage}}{\text{Minimum voltage}} = \frac{\left|\overline{V}_F\right| + \left|\overline{V}_B\right|}{\left|\overline{V}_F\right| - \left|\overline{V}_B\right|}$$

The VSWR can be stated in terms of the reflection coefficient $\overline{\rho}_L$

$$VSWR = \frac{1 + \frac{\left|\overline{V}_B\right|}{\left|\overline{V}_F\right|}}{1 - \frac{\left|\overline{V}_B\right|}{\left|\overline{V}_F\right|}} = \frac{1 + \left|\frac{\overline{V}_B}{\overline{V}_F}\right|}{1 - \left|\frac{\overline{V}_B}{\overline{V}_F}\right|} = \frac{1 + \left|\rho_L\right|}{1 - \left|\rho_L\right|}$$

Or alternatively (and more usefully) the reflection coefficient $\overline{\rho}_L$ can be stated in terms of the VSWR (which can be measured)

$$\left|\rho_L\right| = \frac{VSWR - 1}{VSWR + 1} \qquad (3.2)$$

If there is total reflection then $\overline{\rho}_L$ = 1 and VSWR is infinite.

Zero reflection leads to VSWR=1

I.3.4 Summary

•Full power transfer requires $\overline{\rho}_L = 0$

•When $\overline{\rho}_L = 0$ a load is said to be "matched"

•The advantages of matching are that:

1) We get all the power to the load
2) There are no echoes

•The simplest way to match a line to a load is to set

$$Z_0 = \overline{Z}_L$$

i.e. so that the load equals the characteristic impedance

Since, from (3.1b): $\overline{\rho}_L = \dfrac{\overline{Z}_L - Z_0}{\overline{Z}_L + Z_0}$

•Fraction of power reflected = $\left|\overline{\rho}_L\right|^2$

•Reflections will set up standing waves.

The Voltage Standing Wave Ratio (VSWR) is given by:

$$VSWR = \frac{1+\left|\rho_L\right|}{1-\left|\rho_L\right|}$$