Dynamics of Beams, Plates and Shells
- Handout 6 -

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Completed Version

Strong and Weak Form of Elastodynamics

- Equilibrium equations for elastodynamics (strong form)

$$\underbrace{-\rho \ddot{u}_i}_{\text{inertia}} + \underbrace{\sigma_{ij,j} + b_i}_{\text{same as in statics}} = 0$$

  - Density ρ
  - Acceleration vector $\ddot{u}_i$
  - Stress matrix σ_{ij}
  - Distributed body force vector b_i

  - Displacement initial condition (at time=0) $u_i(t=0) = \overline{u}_i$
  - Velocity initial condition (at time=0) $\dot{u}_i(t=0) = \dot{\overline{u}}_i$

- The weak form of the equilibrium equations for elastodynamics is known as the d'Alembert's principle
  - It can be obtained by the standard procedure: Multiply the strong form with a test function, integrate by parts and apply the divergence theorem

$$\int_\Omega \rho \ddot{u}_i v_i \, d\Omega + \underbrace{\int_\Omega \sigma_{ij} v_{i,j} \, d\Omega = \int_\Omega b_i v_i \, d\Omega}_{\text{same as in statics}}$$

  - Virtual displacement vector v_i
  - Build-in boundaries are assumed so that no boundary integrals are present in the above equation

FE Discretization of Elastodynamics -1-

- The weak form over one typical finite element element Ω_e

$$\int_{\Omega_e} \rho \ddot{u}_i v_i \, d\Omega_e + \underbrace{\int_{\Omega_e} \sigma_{ij} v_{i,j} \, d\Omega_e = \int_{\Omega_e} b_i v_i \, d\Omega_e}_{\text{same as in statics}}$$

  - Approximation of displacements, accelerations and virtual displacements

$$u_i = \sum_K N^K u_i^K \qquad \ddot{u}_i = \sum_K N^K \ddot{u}_i^K \qquad v_i = \sum_K N^K v_i^K$$

    - u_i^K, $\ddot{u}_i^K$ and v_i^K are the nodal displacements, accelerations and virtual displacements, respectively
    - Shape functions are the same as in statics
    - Nodal variables are now a function of time!

  - Element mass matrix is computed by introducing the approximations into kinetic virtual work

$$\int_{\Omega_e} \rho \ddot{u}_i v_i \, d\Omega = \sum_K \sum_L \ddot{u}_i^K v_i^K \int_{\Omega_e} \rho N^K N^L \, d\Omega_e$$

$$\boldsymbol{m}_e = \sum_K \sum_L \int_{\Omega_e} \rho N^K N^L \, d\Omega_e$$

  - Element stiffness matrix and the load vector are the same as for the static case

FE Discretization of Elastodynamics -2-

- Global semi-discrete equation of motion after assembly of element matrices

$$M\ddot{u} + Ku = F$$

  - Global mass matrix M
  - Global stiffness matrix K
  - Global external force vector F
  - Initial conditions $u(0) = \overline{u}, \qquad \dot{u}(0) = \overline{\dot{u}}$

  - Equation called semi-discrete because it is discretized in space but still continuous in time

- Global semi-discrete equation of motion with viscous damping

$$M\ddot{u} + C\dot{u} + Ku = F$$

  - Damping matrix C
  - Damping proportional to velocity

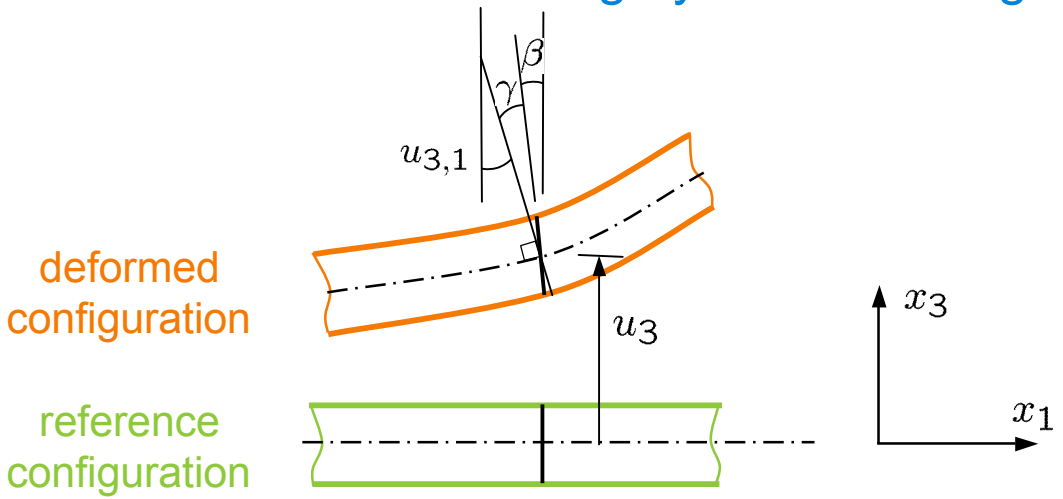
  - Rayleigh damping (widely used in structural engineering)

$$C = aM + bK$$

  - a and b are two scalar structure properties which are determined from experiments

Dynamics of Timoshenko Beams

- Assumed deformations during dynamic loading



- Kinematic assumption: a plane section originally normal to the centroid remains plane, but in addition also shear deformations occur
  - Key kinematic relations for statics: $u_1 = -\beta x_3$ (small rotations assumed)
    $$u_{3,1} = \gamma + \beta$$
  - Corresponding relations for dynamics: $\ddot{u}_1 = -\ddot{\beta} x_3$ (small rotations assumed)
    $$\ddot{u}_{3,1} = \ddot{\gamma} + \ddot{\beta}$$

Kinetic Virtual Work for Timoshenko Beam

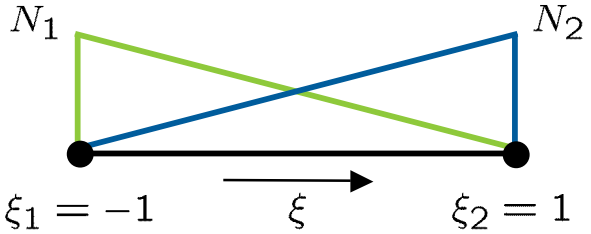
- Introducing the beam accelerations and test functions into the kinetic virtual work of elastodynamics gives

$$\int_\Omega \int_{-t/2}^{t/2} \rho \ddot{u}_i v_i \, dx_3 dx_1 = \int_\Omega \int_{-t/2}^{t/2} \rho \begin{bmatrix} \ddot{u}_1 & \ddot{u}_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} dx_3 dx_1$$
$$= \int_\Omega \int_{-t/2}^{t/2} \rho (\ddot{\beta}\phi x_3^2 + \ddot{u}_3 v_3) \, dx_3 dx_1$$

  - Virtual axial displacements $v_1 = \phi x_3$
  - Virtual deflections and rotations $v_3, \, \phi$
  - Kinetic virtual work due to rotation $\int_\Omega \int_{-t/2}^{t/2} \rho \ddot{\beta} \phi x_3^2 \, dx_3 dx_1 = \int_\Omega \rho I \ddot{\beta} \phi \, dx_1$
    - Rotationary inertia (very small for thin beams) $I = \dfrac{t^3}{12}$
  - Kinetic virtual work due to deflection $\int_\Omega \int_{-t/2}^{t/2} \rho \ddot{u}_3 v_3 \, dx_3 dx_1 = \int_\Omega \rho t \ddot{u}_3 v_3 \, dx_1$

Finite Element Discretization - Mass Matrix

- Interpolation with shape functions (e.g. linear shape functions)



$$N^1(\xi) = \frac{1}{2}(1-\xi)$$
$$N^2(\xi) = \frac{1}{2}(1+\xi)$$

- Consistent mass matrix is computed by introducing the interpolations into the kinetic virtual work

$$\boldsymbol{m}_e = \rho \int_{\Omega_e} \begin{bmatrix} tN_1N_1 & 0 & tN_1N_2 & 0 \\ 0 & IN_1N_1 & 0 & IN_1N_2 \\ tN_2N_1 & 0 & tN_2N_2 & 0 \\ 0 & IN_2N_1 & 0 & IN_2N_2 \end{bmatrix} dx_1 \begin{matrix} \\ \\ \\ \\ u_3^1 \quad \beta^1 \quad u_3^2 \quad \beta^2 \end{matrix}$$

  - For practical computations lumped mass matrix sufficient (the sum of the elements of each row of the consistent mass matrix is used as the diagonal element)

$$\boldsymbol{m}_e = \frac{\rho l_e}{2} \begin{bmatrix} t & 0 & 0 & 0 \\ & I & 0 & 0 \\ & & t & 0 \\ \text{sym.} & & & I \end{bmatrix}$$

    - The components of the mass matrix are simply the total element mass and rotational inertia divided by two

Most Basic Time Integration Scheme -1-

- Semi-discrete equation of motion

$$M\ddot{u} + Ku = F$$

- Time integration
  - Assume displacements, velocities and accelerations are known for $t \le t_n$
  - Central difference formula for the velocity

$$\dot{u}_{n+\frac{1}{2}} = \frac{u_{n+1} - u_n}{\Delta t}$$

  - Central difference formula for the acceleration

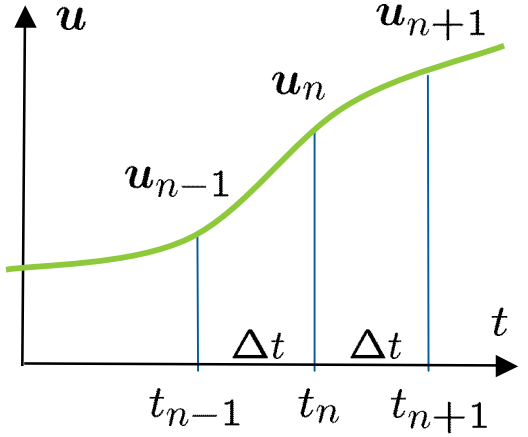
$$\ddot{u}_n = \frac{\dot{u}_{n+\frac{1}{2}} - \dot{u}_{n-\frac{1}{2}}}{\Delta t} = \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta t^2}$$

  - Discrete equilibrium at $t=t_n$

$$M\ddot{u}_n + Ku_n = F_n \rightarrow \ddot{u}_n = M^{-1}(F_n - Ku_n)$$

  - Substituting the computed acceleration into the central difference formula for acceleration yields the displacements at $t=t_{n+1}$

$$u_{n+1} = 2u_n - u_{n-1} + \Delta t^2 M^{-1}(F_n - Ku_n)$$



Most Basic Time Integration Scheme -2-

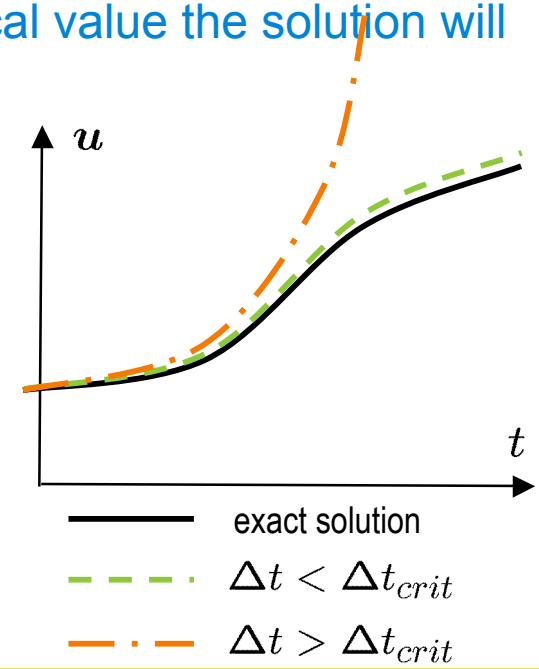
- These equations are repeatedly used in order to march in time and to obtain solutions at times $t=t_{n+2}$, t_{n+3}, ...
- Provided that the mass matrix is diagonal displacements and velocities are computed without inverting any matrices. Such a scheme is called explicit.
  - Matrix inversion is usually the most time consuming part of finite element analysis
  - In most real world applications, explicit time integration schemes are used

- Explicit time integration is very easy to implement. The disadvantage is conditional stability. If the time step exceeds a critical value the solution will grow unboundedly
  - Critical time step size

$$\Delta t_{crit} = \frac{l_e}{c}$$

  - Characteristic element size l_e
  - Wave speed c
  - Longitudinal wave speed in solids (material property)

$$c = \sqrt{\frac{E}{\rho}}$$

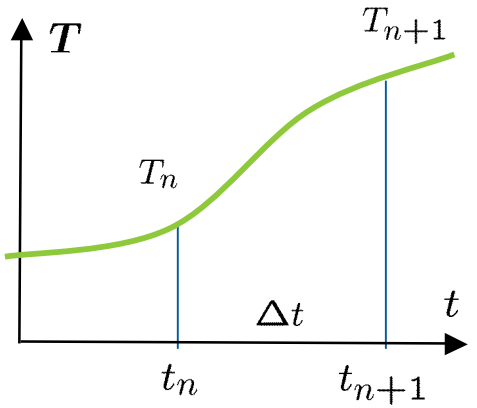


Advanced Time Integration Schemes -1-

- It is instructive to consider the time integration of the semi-discrete heat equation before attempting the time integration for elasticity

$$\boldsymbol{M}\dot{\boldsymbol{T}} + \boldsymbol{K}\boldsymbol{T} = \boldsymbol{F}$$

  - Temperature vector and its time derivative $\boldsymbol{T}$, $\dot{\boldsymbol{T}}$
  - Heat capacity matrix $\boldsymbol{M}$
  - Heat conductivity matrix $\boldsymbol{K}$
  - Heat supply vector $\boldsymbol{F}$
  - Initial conditions $\boldsymbol{T}(t=0) = \overline{\boldsymbol{T}}$



- α - family of time integration schemes

$$\boldsymbol{T}_{n+1} = \boldsymbol{T}_n + \Delta t \dot{\boldsymbol{T}}_{n+\alpha}$$

  - with $\dot{\boldsymbol{T}}_{n+\alpha} = (1-\alpha)\dot{\boldsymbol{T}}_n + \alpha \dot{\boldsymbol{T}}_{n+1}$
  - Combining both gives $\boldsymbol{T}_{n+1} = \boldsymbol{T}_n + \Delta t \left[(1-\alpha)\dot{\boldsymbol{T}}_n + \alpha \dot{\boldsymbol{T}}_{n+1}\right]$
  - Introducing into the semidiscrete heat equation gives an equation for determining $\dot{\boldsymbol{T}}_{n+1}$

$$\boldsymbol{M}\dot{\boldsymbol{T}}_{n+1} + \boldsymbol{K}\boldsymbol{T}_{n+1} = \boldsymbol{F}_{n+1}$$

Advanced Time Integration Schemes -2-

- Common names for the resulting methods
  - $\alpha = 0$ forward differences; forward Euler
  - $\alpha = 1/2$ trapezoidal rule; midpoint rule; Crank-Nicholson
  - $\alpha = 1$ backward differences; backward Euler

- Explicit vs. implicit methods and their stability
  - The method is explicit and conditionally stable for $\alpha = 0$
    - Time step size restricted
  - The method is implicit and stable for $\alpha \neq 0$
    - Time step size not restricted. However, for large time steps less accurate.

- Implementation in a predictor-corrector form
  - Simplifies the computer implementation. Does not change the basic method.
  - Compute a predictor $\tilde{\boldsymbol{T}}_{n+1}$ with known solution

$$\tilde{\boldsymbol{T}}_{n+1} = \boldsymbol{T}_n + (1-\alpha)\Delta t \dot{\boldsymbol{T}}_n \qquad \Rightarrow \boldsymbol{T}_{n+1} = \tilde{\boldsymbol{T}}_{n+1} + \alpha \Delta t \dot{\boldsymbol{T}}_{n+1}$$

  - Introducing into the semidiscrete heat equation gives an equation for determining $\dot{\boldsymbol{T}}_{n+1}$

$$(\boldsymbol{M} + \alpha \boldsymbol{K}) \dot{\boldsymbol{T}}_{n+1} = \boldsymbol{F}_{n+1} - \boldsymbol{K} \tilde{\boldsymbol{T}}_{n+1}$$

Newmark's Scheme -1-

- Newmark family of time integration schemes are widely used in structural dynamics
  - Assume that the displacements, velocities, and accelerations are known for $t \leq t_n$
  - Displacements and velocities are approximated with

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \Delta t \dot{\boldsymbol{u}}_n + \frac{\Delta t^2}{2}[(1-2\beta)\ddot{\boldsymbol{u}}_n + 2\beta \ddot{\boldsymbol{u}}_{n+1}]$$

$$\dot{\boldsymbol{u}}_{n+1} = \dot{\boldsymbol{u}}_n + \Delta t[(1-\gamma)\ddot{\boldsymbol{u}}_n + \gamma \ddot{\boldsymbol{u}}_{n+1}]$$

    - with two scalar parameters $\gamma,\ \beta$
    - The two scalar parameters determine the accuracy of the scheme
    - Unconditionally stable and undamped for $\beta = \frac{1}{4}\ \gamma = \frac{1}{2}$

- Implementation in a-form (according to Hughes)
  - Simplifies the computer implementation. Does not change the basic method.
  - Compute predictor velocities and displacements $\tilde{\boldsymbol{u}}_{n+1},\ \tilde{\dot{\boldsymbol{u}}}_{n+1}$

$$\tilde{\boldsymbol{u}}_{n+1} = \boldsymbol{u}_n + \Delta t \dot{\boldsymbol{u}}_n + \frac{\Delta t^2}{2}(1-2\beta)\ddot{\boldsymbol{u}}_n$$

$$\tilde{\dot{\boldsymbol{u}}}_{n+1} = \dot{\boldsymbol{u}}_n + \Delta t(1-\gamma)\ddot{\boldsymbol{u}}_n$$

Newmark's Scheme -2-

- Displacement and velocity approximation using predictor values

$$\boldsymbol{u}_{n+1} = \tilde{\boldsymbol{u}}_{n+1} + \beta \Delta t^2 \ddot{\boldsymbol{u}}_{n+1}$$

$$\dot{\boldsymbol{u}}_{n+1} = \tilde{\dot{\boldsymbol{u}}}_{n+1} + \gamma \Delta t \ddot{\boldsymbol{u}}_{n+1}$$

- Introducing the displacement approximation into the semidiscrete equation gives an equation for computing the new accelerations at time $t=t_{n+1}$

$$\boldsymbol{M}\ddot{\boldsymbol{u}}_{n+1} + \boldsymbol{K}\boldsymbol{u}_{n+1} = \boldsymbol{F}_{n+1}$$

$$(\boldsymbol{M} + \beta \Delta t^2 \boldsymbol{K})\ddot{\boldsymbol{u}}_{n+1} = \boldsymbol{F}_{n+1} - \boldsymbol{K}\tilde{\boldsymbol{u}}_{n+1}$$

  - The new displacements and velocities are computed from the displacement and velocity approximation equations