

Dynamics of Beams, Plates and Shells - Handout 6 -

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Completed Version

Strong and Weak Form of Elastodynamics

■ Equilibrium equations for elastodynamics (strong form)

$$\underbrace{-\rho \ddot{u}_i}_{\text{inertia}} + \underbrace{\sigma_{ij,j} + b_i}_{\text{same as in statics}} = 0$$

- Density ρ
- Acceleration vector \ddot{u}_i
- Stress matrix σ_{ij}
- Distributed body force vector b_i

- Displacement initial condition (at time=0) $u_i(t = 0) = \bar{u}_i$
- Velocity initial condition (at time=0) $\dot{u}_i(t = 0) = \bar{\dot{u}}_i$

■ The weak form of the equilibrium equations for elastodynamics is known as the d'Alembert's principle

- It can be obtained by the standard procedure: Multiply the strong form with a test function, integrate by parts and apply the divergence theorem

$$\int_{\Omega} \rho \ddot{u}_i v_i d\Omega + \underbrace{\int_{\Omega} \sigma_{ij} v_{i,j} d\Omega}_{\text{same as in statics}} = \int_{\Omega} b_i v_i d\Omega$$

- Virtual displacement vector v_i
- Build-in boundaries are assumed so that no boundary integrals are present in the above equation

FE Discretization of Elastodynamics -1-

- The weak form over one typical finite element element Ω_e

$$\int_{\Omega_e} \rho \ddot{u}_i v_i d\Omega_e + \underbrace{\int_{\Omega_e} \sigma_{ij} v_{i,j} d\Omega_e}_{\text{same as in statics}} = \int_{\Omega_e} b_i v_i d\Omega_e$$

- Approximation of displacements, accelerations and virtual displacements

$$u_i = \sum_K N^K u_i^K \quad \ddot{u}_i = \sum_K N^K \ddot{u}_i^K \quad v_i = \sum_K N^K v_i^K$$

- u_i^K , \ddot{u}_i^K and v_i^K are the nodal displacements, accelerations and virtual displacements, respectively
- Shape functions are the same as in statics
- Nodal variables are now a function of time!

- Element mass matrix is computed by introducing the approximations into kinetic virtual work

$$\int_{\Omega_e} \rho \ddot{u}_i v_i d\Omega = \sum_K \sum_L \ddot{u}_i^K v_i^K \int_{\Omega_e} \rho N^K N^L d\Omega_e$$

$$m_e = \sum_K \sum_L \int_{\Omega_e} \rho N^K N^L d\Omega_e$$

- Element stiffness matrix and the load vector are the same as for the static case

FE Discretization of Elastodynamics -2-

- Global semi-discrete equation of motion after assembly of element matrices

$$M\ddot{u} + Ku = F$$

- Global mass matrix M
- Global stiffness matrix K
- Global external force vector F
- Initial conditions $u(0) = \bar{u}, \quad \dot{u}(0) = \bar{\dot{u}}$

- Equation called semi-discrete because it is discretized in space but still continuous in time

- Global semi-discrete equation of motion with viscous damping

$$M\ddot{u} + C\dot{u} + Ku = F$$

- Damping matrix C
- Damping proportional to velocity

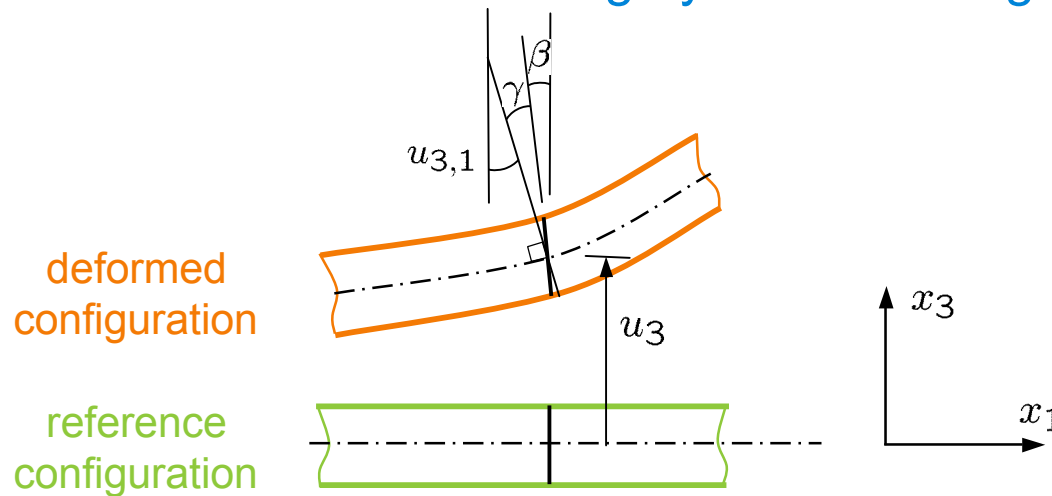
- Rayleigh damping (widely used in structural engineering)

$$C = aM + bK$$

- a and b are two scalar structure properties which are determined from experiments

Dynamics of Timoshenko Beams

- Assumed deformations during dynamic loading



- Kinematic assumption: a plane section originally normal to the centroid remains plane, but in addition also shear deformations occur

- Key kinematic relations for statics: $u_1 = -\beta x_3$ (small rotations assumed)

$$u_{3,1} = \gamma + \beta$$

- Corresponding relations for dynamics: $\ddot{u}_1 = -\ddot{\beta} x_3$ (small rotations assumed)

$$\ddot{u}_{3,1} = \ddot{\gamma} + \ddot{\beta}$$

Kinetic Virtual Work for Timoshenko Beam

- Introducing the beam accelerations and test functions into the kinetic virtual work of elastodynamics gives

$$\begin{aligned}\int_{\Omega} \int_{-t/2}^{t/2} \rho \ddot{u}_i v_i dx_3 dx_1 &= \int_{\Omega} \int_{-t/2}^{t/2} \rho \begin{bmatrix} \ddot{u}_1 & \ddot{u}_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} dx_3 dx_1 \\ &= \int_{\Omega} \int_{-t/2}^{t/2} \rho (\ddot{\beta} \phi x_3^2 + \ddot{u}_3 v_3) dx_3 dx_1\end{aligned}$$

- Virtual axial displacements $v_1 = \phi x_3$
- Virtual deflections and rotations v_3, ϕ

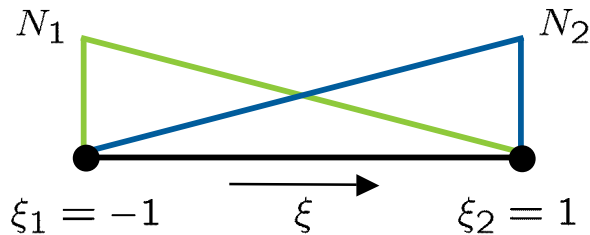
- Kinetic virtual work due to rotation $\int_{\Omega} \int_{-t/2}^{t/2} \rho \ddot{\beta} \phi x_3^2 dx_3 dx_1 = \int_{\Omega} \rho I \ddot{\beta} \phi dx_1$

- Rotationaly inertia (very small for thin beams) $I = \frac{t^3}{12}$

- Kinetic virtual work due to deflection $\int_{\Omega} \int_{-t/2}^{t/2} \rho \ddot{u}_3 v_3 dx_3 dx_1 = \int_{\Omega} \rho t \ddot{u}_3 v_3 dx_1$

Finite Element Discretization - Mass Matrix

- Interpolation with shape functions (e.g. linear shape functions)



$$N^1(\xi) = \frac{1}{2}(1 - \xi)$$

$$N^2(\xi) = \frac{1}{2}(1 + \xi)$$

- Consistent mass matrix is computed by introducing the interpolations into the kinetic virtual work

$$m_e = \rho \int_{\Omega_e} \begin{bmatrix} tN_1N_1 & 0 & tN_1N_2 & 0 \\ 0 & IN_1N_1 & 0 & IN_1N_2 \\ tN_2N_1 & 0 & tN_2N_2 & 0 \\ 0 & IN_2N_1 & 0 & IN_2N_2 \end{bmatrix} dx_1$$

$$\begin{matrix} u_3^1 & \beta^1 & u_3^2 & \beta^2 \end{matrix}$$

- For practical computations lumped mass matrix sufficient (the sum of the elements of each row of the consistent mass matrix is used as the diagonal element)

$$m_e = \frac{\rho l_e}{2} \begin{bmatrix} t & 0 & 0 & 0 \\ & I & 0 & 0 \\ & & t & 0 \\ \text{sym.} & & & I \end{bmatrix}$$

- The components of the mass matrix are simply the total element mass and rotational inertia divided by two

Most Basic Time Integration Scheme -1-

- Semi-discrete equation of motion

$$M\ddot{u} + Ku = F$$

- Time integration

- Assume displacements, velocities and accelerations are known for $t \leq t_n$
- Central difference formula for the velocity

$$\dot{u}_{n+\frac{1}{2}} = \frac{u_{n+1} - u_n}{\Delta t}$$

- Central difference formula for the acceleration

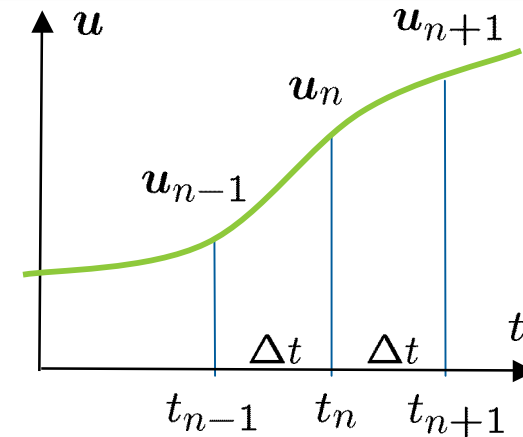
$$\ddot{u}_n = \frac{\dot{u}_{n+\frac{1}{2}} - \dot{u}_{n-\frac{1}{2}}}{\Delta t} = \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta t^2}$$

- Discrete equilibrium at $t=t_n$

$$M\ddot{u}_n + Ku_n = F_n \rightarrow \ddot{u}_n = M^{-1}(F_n - Ku_n)$$

- Substituting the computed acceleration into the central difference formula for acceleration yields the displacements at $t=t_{n+1}$

$$u_{n+1} = 2u_n - u_{n-1} + \Delta t^2 M^{-1}(F_n - Ku_n)$$



Most Basic Time Integration Scheme -2-

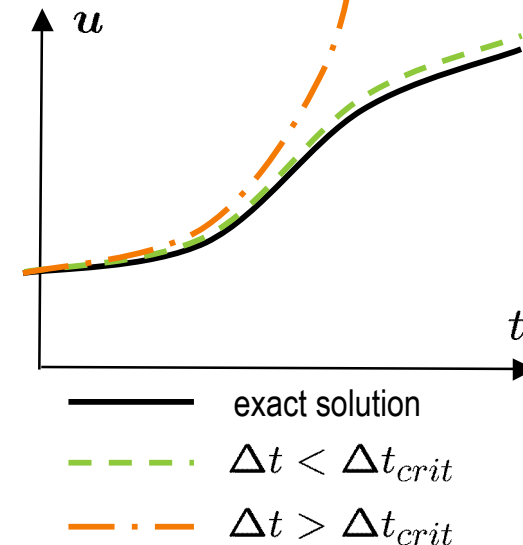
- These equations are repeatedly used in order to march in time and to obtain solutions at times $t=t_{n+2}, t_{n+3}, \dots$
- Provided that the mass matrix is diagonal displacements and velocities are computed without inverting any matrices. Such a scheme is called explicit.
 - Matrix inversion is usually the most time consuming part of finite element analysis
 - In most real world applications, explicit time integration schemes are used
- **Explicit time integration is very easy to implement. The disadvantage is conditional stability. If the time step exceeds a critical value the solution will grow unboundedly**

- Critical time step size

$$\Delta t_{crit} = \frac{l_e}{c}$$

- Characteristic element size l_e
- Wave speed c
- Longitudinal wave speed in solids (material property)

$$c = \sqrt{\frac{E}{\rho}}$$



Advanced Time Integration Schemes -1-

- It is instructive to consider the time integration of the semi-discrete heat equation before attempting the time integration for elasticity

$$M\dot{T} + KT = F$$

- Temperature vector and its time derivative T, \dot{T}
- Heat capacity matrix M
- Heat conductivity matrix K
- Heat supply vector F
- Initial conditions $T(t = 0) = \bar{T}$

- α - family of time integration schemes

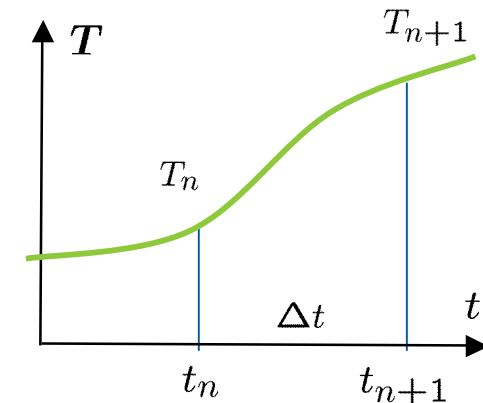
$$T_{n+1} = T_n + \Delta t \dot{T}_{n+\alpha}$$

- with $\dot{T}_{n+\alpha} = (1 - \alpha)\dot{T}_n + \alpha\dot{T}_{n+1}$

- Combining both gives $T_{n+1} = T_n + \Delta t [(1 - \alpha)\dot{T}_n + \alpha\dot{T}_{n+1}]$

- Introducing into the semidiscrete heat equation gives an equation for determining \dot{T}_{n+1}

$$M\dot{T}_{n+1} + KT_{n+1} = F_{n+1}$$



Advanced Time Integration Schemes -2-

- Common names for the resulting methods
 - $\alpha = 0$ forward differences; forward Euler
 - $\alpha = 1/2$ trapezoidal rule; midpoint rule; Crank-Nicholson
 - $\alpha = 1$ backward differences; backward Euler
- Explicit vs. implicit methods and their stability
 - The method is explicit and conditionally stable for $\alpha = 0$
 - Time step size restricted
 - The method is implicit and stable for $\alpha \neq 0$
 - Time step size not restricted. However, for large time steps less accurate.
- Implementation in a predictor-corrector form
 - Simplifies the computer implementation. Does not change the basic method.
 - Compute a predictor \tilde{T}_{n+1} with known solution
$$\tilde{T}_{n+1} = T_n + (1 - \alpha)\Delta t \dot{T}_n \quad \Rightarrow \quad T_{n+1} = \tilde{T}_{n+1} + \alpha\Delta t \dot{T}_{n+1}$$
 - Introducing into the semidiscrete heat equation gives an equation for determining \dot{T}_{n+1}
$$(M + \alpha K)\dot{T}_{n+1} = F_{n+1} - K\tilde{T}_{n+1}$$

Newmark's Scheme -1-

- Newmark family of time integration schemes are widely used in structural dynamics

- Assume that the displacements, velocities, and accelerations are known for $t \leq t_n$
- Displacements and velocities are approximated with

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \dot{\mathbf{u}}_n + \frac{\Delta t^2}{2} [(1 - 2\beta)\ddot{\mathbf{u}}_n + 2\beta\ddot{\mathbf{u}}_{n+1}]$$

$$\dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \Delta t [(1 - \gamma)\ddot{\mathbf{u}}_n + \gamma\ddot{\mathbf{u}}_{n+1}]$$

- with two scalar parameters γ, β
- The two scalar parameters determine the accuracy of the scheme
- Unconditionally stable and undamped for $\beta = \frac{1}{4}, \gamma = \frac{1}{2}$

- Implementation in a-form (according to Hughes)

- Simplifies the computer implementation. Does not change the basic method.
- Compute predictor velocities and displacements $\tilde{\mathbf{u}}_{n+1}, \tilde{\dot{\mathbf{u}}}_{n+1}$

$$\tilde{\mathbf{u}}_{n+1} = \mathbf{u}_n + \Delta t \dot{\mathbf{u}}_n + \frac{\Delta t^2}{2} (1 - 2\beta)\ddot{\mathbf{u}}_n$$

$$\tilde{\dot{\mathbf{u}}}_{n+1} = \dot{\mathbf{u}}_n + \Delta t (1 - \gamma)\ddot{\mathbf{u}}_n$$

Newmark's Scheme -2-

- Displacement and velocity approximation using predictor values

$$\mathbf{u}_{n+1} = \tilde{\mathbf{u}}_{n+1} + \beta \Delta t^2 \ddot{\mathbf{u}}_{n+1}$$

$$\dot{\mathbf{u}}_{n+1} = \tilde{\dot{\mathbf{u}}}_{n+1} + \gamma \Delta t \ddot{\mathbf{u}}_{n+1}$$

- Introducing the displacement approximation into the semidiscrete equation gives an equation for computing the new accelerations at time $t=t_{n+1}$

$$\mathbf{M} \ddot{\mathbf{u}}_{n+1} + \mathbf{K} \mathbf{u}_{n+1} = \mathbf{F}_{n+1}$$

$$(\mathbf{M} + \beta \Delta t^2 \mathbf{K}) \ddot{\mathbf{u}}_{n+1} = \mathbf{F}_{n+1} - \mathbf{K} \tilde{\mathbf{u}}_{n+1}$$

- The new displacements and velocities are computed from the displacement and velocity approximation equations