Finite Element Formulation for Shells
- Handout 5 -

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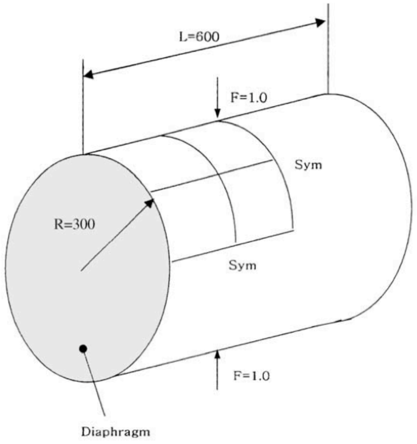
Completed Version

Overview of Shell Finite Elements

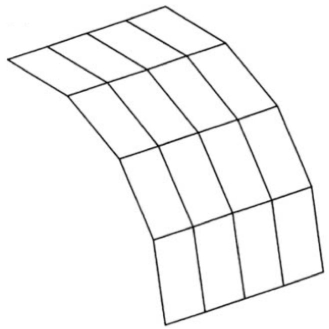
- There are three different approaches for deriving shell finite elements
  - Flat shell elements
    - The geometry of a shell is approximated with flat finite elements
    - Flat shell elements are obtained by combining plate elements with plate stress elements
  - “Degenerated” shell elements
    - Elements are derived by “degenerating” a three dimensional solid finite element into a shell surface element

Flat Shell Finite Elements

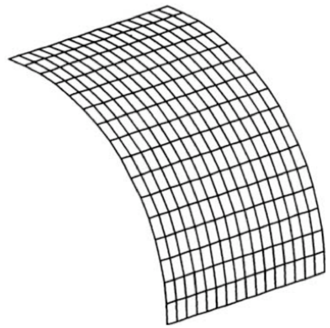
- Example: Discretization of a cylindrical shell with flat shell finite elements



Cylindrical shell



Coarse mesh



Fine mesh

  - Note that due to symmetry only one eight of the shell is discretized

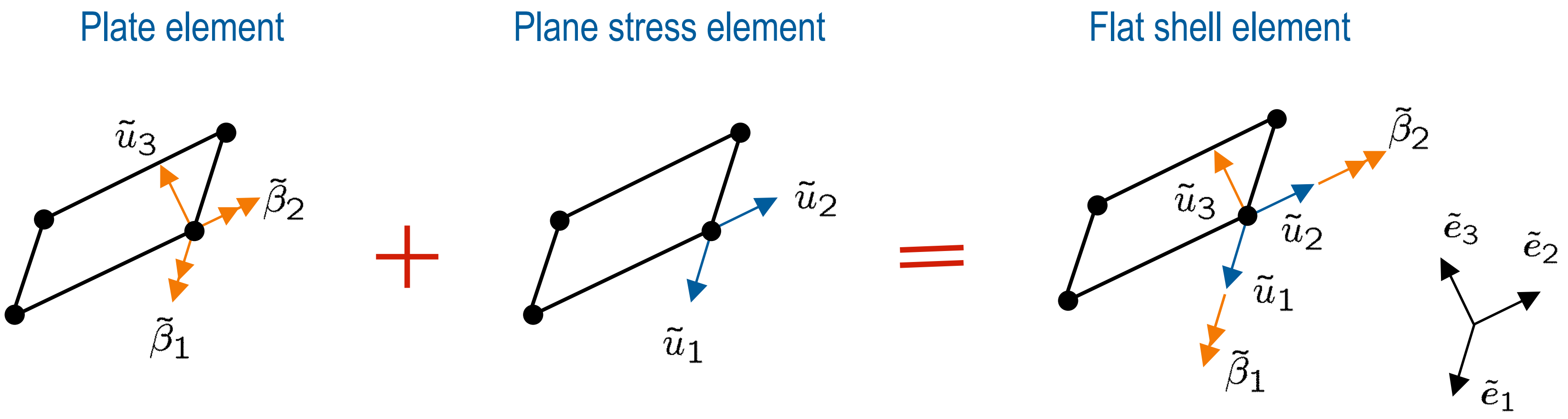
- The quality of the surface approximation improves if more and more flat elements are used

- Flat shell finite elements are derived by superposition of plate finite elements with plane stress finite elements
  - As plate finite elements usually Reissner-Mindlin plate elements are used
  - As plane stress elements the finite elements derived in 3D7 are used

  - Overall approach equivalent to deriving frame finite elements by superposition of beam and truss finite elements

Four-Noded Flat Shell Element -1-

- First the degrees of freedom of a plate and plane-stress finite element in a local element-aligned coordinate system are considered



- The local base vectors $\tilde{e}_1$ and $\tilde{e}_2$ are in the plane of the element and $\tilde{e}_3$ is orthogonal to the element

- The plate element has three degrees of freedom per node (one out-of-plane displacement and two rotations)
- The plane stress element has two degrees of freedom per node node (two in plane displacements)
- The resulting flat shell element has five degrees of freedom per node

Four-Noded Flat Shell Element -2-

- Stiffness matrix of the plate in the local coordinate system: $\underbrace{\tilde{\boldsymbol{k}}_{ben}}_{12\times 12 \text{ matrix}}$
- Stiffness matrix of the plane stress element in the local coordinate system: $\underbrace{\tilde{\boldsymbol{k}}_{mem}}_{8\times 8 \text{ matrix}}$
- Stiffness matrix of the flat shell element in the local coordinate system

$$\underbrace{\tilde{\boldsymbol{k}}}_{20\times 20 \text{ matrix}} = \begin{bmatrix} \tilde{\boldsymbol{k}}_{mem} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{k}}_{ben} \end{bmatrix}$$

  - Stiffness matrix of the flat shell element can be augmented to include the rotations $\tilde{\beta}_3$ (see figure on previous page)

$$\underbrace{\tilde{\boldsymbol{k}}^*}_{24\times 24 \text{ matrix}} = \begin{bmatrix} \tilde{\boldsymbol{k}}_{mem} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{k}}_{ben} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

    - Stiffness components corresponding to $\tilde{\beta}_3$ are zero because neither the plate nor the plane stress element has corresponding stiffness components

Four-Noded Flat Shell Element -3-

- Transformation of the element stiffness matrix from the local to the global coordinate system
  - Discrete element equilibrium equation in the local coordinate system

    $$\tilde{\boldsymbol{k}}^* \tilde{\boldsymbol{u}} = \tilde{\boldsymbol{f}}$$

    - Nodal displacements and rotations of element $\tilde{\boldsymbol{u}}$
    - Element force vector $\tilde{\boldsymbol{f}}$
  - Transformation of vectors from the local to the global coordinate system

    $$\boldsymbol{u} = \boldsymbol{c}\tilde{\boldsymbol{u}} \qquad \boldsymbol{f} = \boldsymbol{c}\tilde{\boldsymbol{f}}$$

    - Rotation matrix (or also known as the direction cosine matrix) $\boldsymbol{c}$
    - Note that for all rotation matrices $\boldsymbol{c}^{-1} = \boldsymbol{c}^T$
  - Transformation of element stiffness matrix from the local to global coordinate system

    $$\tilde{\boldsymbol{k}}^* \tilde{\boldsymbol{u}} = \tilde{\boldsymbol{f}} \quad \Rightarrow \quad \tilde{\boldsymbol{k}}^* \boldsymbol{c}^T \boldsymbol{u} = \boldsymbol{c}^T \boldsymbol{f} \quad \Rightarrow \quad \underbrace{\boldsymbol{c}\tilde{\boldsymbol{k}}^* \boldsymbol{c}^T}_{\boldsymbol{k}^*} \boldsymbol{u} = \boldsymbol{f}$$

  - Discrete element equilibrium equation in the global coordinate system

    $$\boldsymbol{k}^* \boldsymbol{u} = \boldsymbol{f}$$

Four-Noded Flat Shell Element -4-

- The global stiffness matrix for the shell structure is constructred by transforming each element matrix into the global coordinate system prior to assembly

- The global force vector of the shell structure is constructed by transforming each element force vector into the global coordinate system prior to assembly

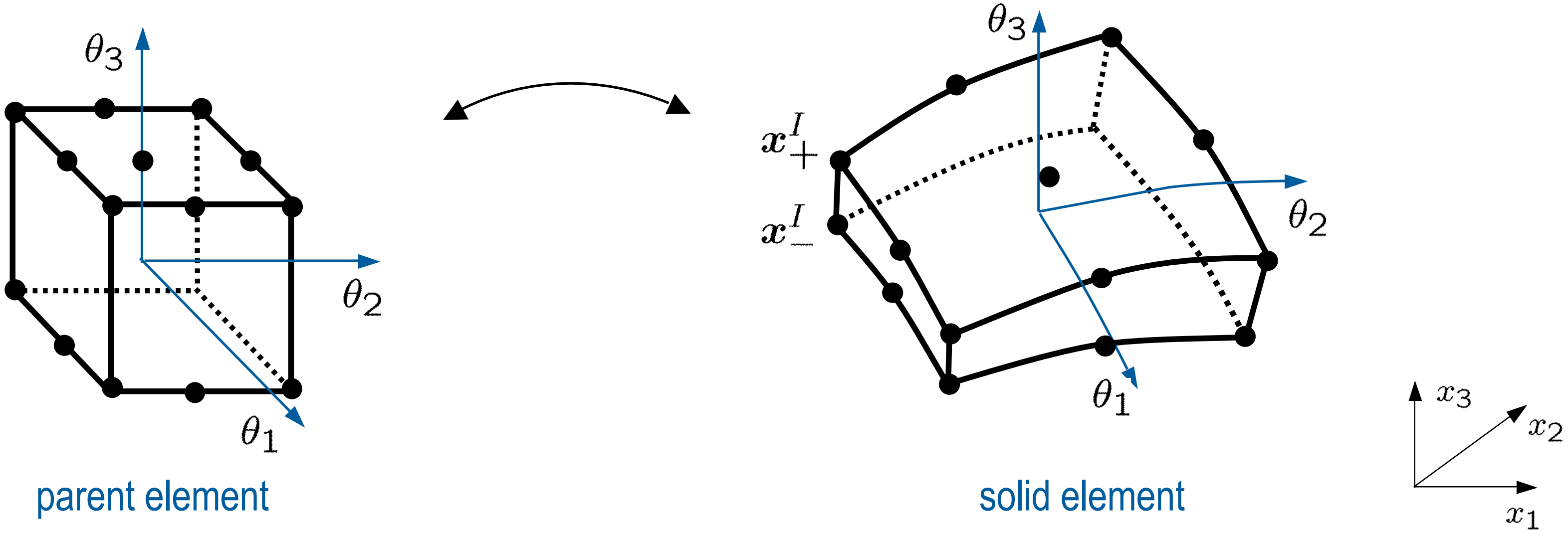
- Remember that there was no stiffness associated with the local rotation degrees of freedom $\tilde{\beta}_3$. Therefore, the global stiffness matrix will be rank deficient if all elements are coplanar.
  - It is possible to add some small stiffness for element stiffness components corresponding to $\tilde{\beta}_3$ in order to make global stiffness matrix invertible

$$\underbrace{\tilde{\boldsymbol{k}}^*}_{24\times 24 \text{ matrix}} = \begin{bmatrix} \tilde{\boldsymbol{k}}_{mem} & 0 & 0 \\ 0 & \tilde{\boldsymbol{k}}_{ben} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Add small stiffness in order to make stiffness matrix invertible

Degenerated Shell Elements -1-

- First a three-dimensional solid element and the corresponding parent element are considered (isoparametric mapping)



- In the following it is assumed that the solid element has on its top and bottom surfaces nine nodes so that the total number of nodes is eighteen
  - The derivations can easily be generalised to arbitrary number of nodes
  - Coordinates of the nodes on the top surface are $\boldsymbol{x}_+^I$ for $I = 1, \cdots, 9$
  - Coordinates of the nodes on the bottom surface are $\boldsymbol{x}_-^I$ for $I = 1, \cdots, 9$

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Degenerated Shell Elements -2-

- There are nine isoparametric shape functions for interpolating the top and bottom surfaces

$$N^I(\theta^1, \theta^2) \text{ for } I = 1, \cdots, 9$$

  - with the natural coordinates $\theta^1, \theta^2 \in (-1, +1)$
  - Note that these shape functions are identical to the ones for two dimensional elasticity

- The geometry of the solid element can be interpolate with

$$\boldsymbol{x}(\theta_1, \theta_2, \theta_3) = \sum_{I=1}^{9} N^I(\theta_1, \theta_2)\frac{1+\theta_3}{2}\boldsymbol{x}_+^I + \sum_{I=1}^{9} N^I(\theta_1, \theta_2)\frac{1-\theta_3}{2}\boldsymbol{x}_-^I$$

$$\text{with } -1 \leq \theta_3 \leq 1$$

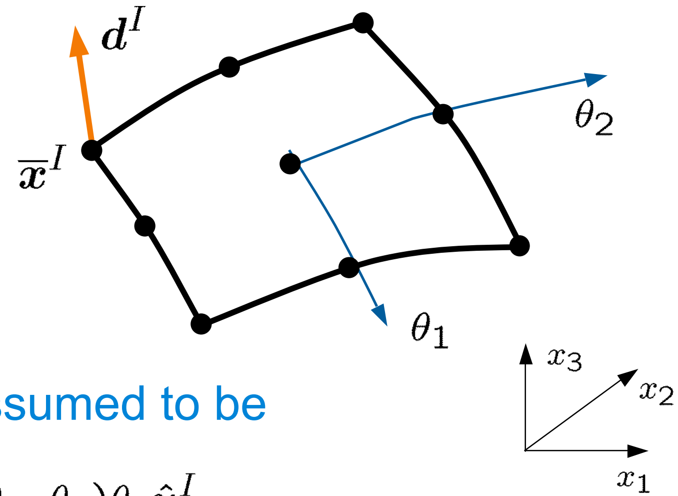
- Definitions
  - Shell mid-surface node $\qquad \overline{\boldsymbol{x}}^I = \frac{1}{2}(\boldsymbol{x}_+^I + \boldsymbol{x}_-^I)$
  - Shell director (or fibre) at node I $\qquad \boldsymbol{d}^I = \frac{\boldsymbol{x}_+^I - \boldsymbol{x}_-^I}{||\boldsymbol{x}_+^I - \boldsymbol{x}_-^I||}$
    - The shell director is a unit vector and is approximately orthogonal to the mid-surface

Degenerated Shell Elements -3-

- Using the previous definitions the solid element geometry can be interpolated with

$$\boldsymbol{x}(\theta_1, \theta_2, \theta_3) = \sum_{I=1}^{9} N^I(\theta_1, \theta_2)\overline{\boldsymbol{x}}^I + \frac{t}{2}\sum_{I=1}^{9} N^I(\theta_1, \theta_2)\theta_3 \boldsymbol{d}^I$$

  - with the solid element thickness t



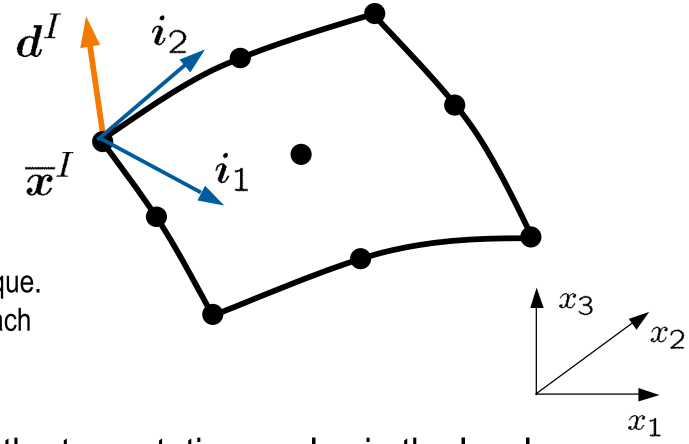
- The displacements of the solid element are assumed to be

$$\boldsymbol{u}(\theta_1, \theta_2, \theta_3) = \sum_{I=1}^{9} N^I(\theta_1, \theta_2)\overline{\boldsymbol{u}}^I + \frac{t}{2}\sum_{I=1}^{9} N^I(\theta_1, \theta_2)\theta_3 \hat{\boldsymbol{u}}^I$$

  - The first component is the mid-surface displacement and the second component is the director displacement
    - Note that the deformed mid-surface nodal coordinates can be computed with $\overline{\boldsymbol{x}}^I + \overline{\boldsymbol{u}}^I$ and the deformed nodal director with $\boldsymbol{d}^I + \hat{\boldsymbol{u}}^I$
  - The director displacement $\hat{\boldsymbol{u}}^I$ has to be constructed so that the director $\boldsymbol{d}^I$ can rotate but not stretch
    - This was one of the of the Reissner-Mindlin theory assumptions

Degenerated Shell Elements -4-

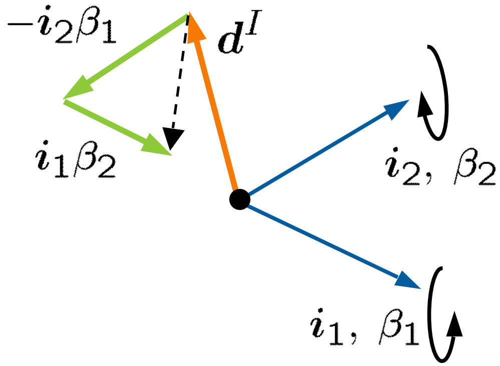
- The director displacements are expressed in terms of rotations at the nodes
  - To accomplish this a local orthonormal coordinate system $\boldsymbol{i}_1 - \boldsymbol{i}_2$ is constructed at each node
    - $\boldsymbol{i}_1 \cdot \boldsymbol{i}_2 = 0$
    - $\boldsymbol{i}_1 \cdot \boldsymbol{i}_1 = 1$ and $\boldsymbol{i}_2 \cdot \boldsymbol{i}_2 = 1$
    - $\boldsymbol{i}_1 \cdot \boldsymbol{d}^I = 0$ and $\boldsymbol{i}_2 \cdot \boldsymbol{d}^I = 0$
      - The definition of the orthonormal coordinate system in not unique. In a finite element implementation it is necessary to store at each node the established coordinate base vectors $\boldsymbol{i}_1 - \boldsymbol{i}_2$.



  - The relationship between the director displacements and the two rotation angles in the local coordinate system is

$$\hat{\boldsymbol{u}}^I = -\boldsymbol{i}_2 \beta_1^I + \boldsymbol{i}_1 \beta_2^I$$

    - Rotations β_1 and β_2 are defined as positive with the right-hand rule
    - It is assumed that the rotation angles are small
    - $\hat{\boldsymbol{u}}^I \cdot \boldsymbol{d}^I = 0$ so that the director length does not change



Degenerated Shell Elements -5-

- Displacement of the shell element in dependence of the mid-surface displacements and director rotations

$$\boldsymbol{u}(\theta_1, \theta_2, \theta_3) = \sum_{I=1}^{9} N^I \overline{\boldsymbol{u}}^I + \frac{t}{2} \sum_{I=1}^{9} N^I \theta_3 (-\boldsymbol{i}_2 \beta_1^I + \boldsymbol{i}_1 \beta_2^I)$$

  - The element has nine nodes
  - There are five unknowns per node (three mid-surface displacements and two director rotations)
  - This assumption about the possible displacements is equivalent to the Reissner-Mindlin assumption

- Introducing the displacements into the strain equation of three-dimensional elasticity leads to the strains of the shell element

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

  - In computing the displacement derivatives the chain rule needs to be used

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial \theta_k}\frac{\partial \theta_k}{\partial x_j} \qquad \text{with} \quad \frac{\partial \theta_k}{\partial x_j} = \left(\frac{\partial x_j}{\partial \theta_k}\right)^{-1}$$

Jacobian

Degenerated Shell Elements -6-

- The Jacobian is computed from the geometry interpolation

$$\boldsymbol{x}(\theta_1, \theta_2, \theta_3) = \sum_{I=1}^{9} N^I(\theta_1, \theta_2)\overline{\boldsymbol{x}}^I + \frac{t}{2}\sum_{I=1}^{9} N^I(\theta_1, \theta_2)\theta_3 \boldsymbol{d}^I$$

$$\frac{\partial \boldsymbol{x}}{\partial \theta_\alpha} = \sum_{I=1}^{9} \frac{\partial N^I}{\partial \theta_\alpha}\overline{\boldsymbol{x}}^I + \frac{t}{2}\sum_{I=1}^{9} \frac{\partial N^I}{\partial \theta_\alpha}\theta_3 \boldsymbol{d}^I \qquad \text{for } \alpha = 1, 2$$

$$\frac{\partial \boldsymbol{x}}{\partial \theta_3} = \frac{t}{2}\sum_{I=1}^{9} N^I \boldsymbol{d}^I$$

- The shell strains introduced into the internal virtual work of three-dimensional elasticity give the internal virtual work of the shell

- For shear locking similar techniques such as developed for the Reissner-Mindlin plate need to be considered