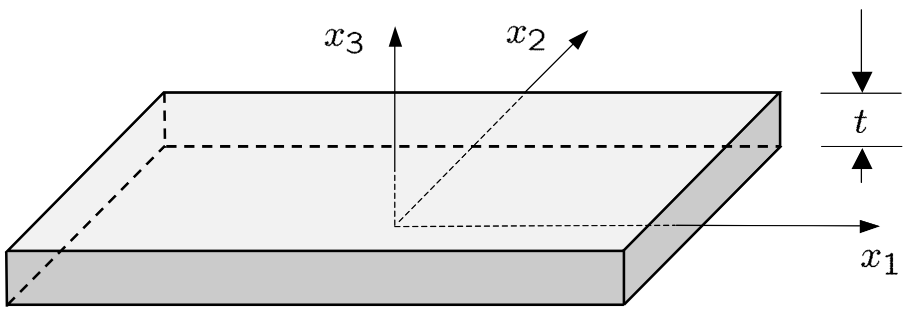
Finite Element Formulation for Plates
- Handout 4 -

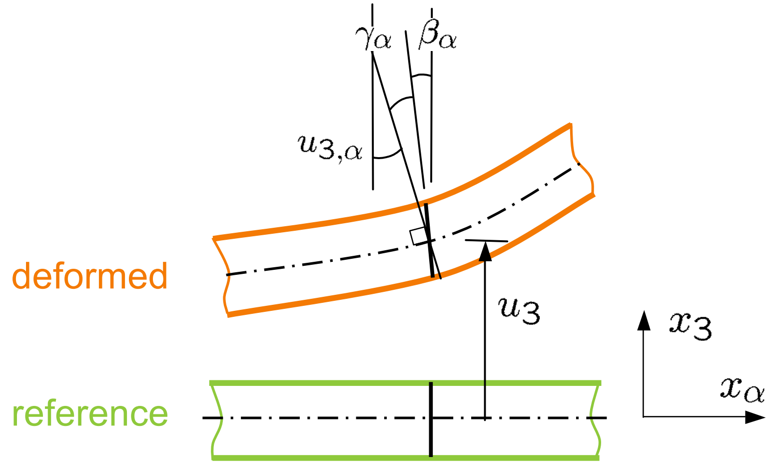
Dr Fehmi Cirak (fc286@)

Completed Version

Kinematics of Reissner-Mindlin Plate -1-

- The extension of Timoshenko beam theory to plates is the Reissner-Mindlin plate theory
- In Reissner-Mindlin plate theory the out-of-plane shear deformations are non-zero (in contrast to Kirchhoff plate theory)
- Almost all commercial codes (Abaqus, LS-Dyna, Ansys, …) use Reissner-Mindlin type plate finite elements
- Assumed displacements during loading





undeformed and deformed geometries along one of the coordinate axis

- Kinematic assumption: a plane section originally normal to the mid-surface remains plane, but in addition also shear deformations occur

Kinematics of Reissner-Mindlin Plate -2-

- Kinematic equations
  - In plane-displacements

$$u_\alpha(x_1, x_2, x_3) = -\beta_\alpha(x_1, x_2)x_3 \quad \text{with } -\frac{t}{2} \leq x_3 \leq \frac{t}{2}, \qquad \alpha = 1, 2$$

    - In this equation and in following all Greek indices only take values 1 or 2
    - It is assumed that rotations are small $(\sin(\beta_\alpha) \approx \beta_\alpha)$
    - Rotation angle of normal: β_α
    - Angle of shearing: γ_α
    - Slope of midsurface: $u_{3,\alpha} = \gamma_\alpha + \beta_\alpha$

  - Out-of-plane displacements

$$u_3(x_1, x_2, x_3) = u_3(x_1, x_2)$$

- The independent variables of the Reissner-Mindlin plate theory are the rotation angle β_α and mid-surface displacement u_3

- Introducing the displacements into the strain equation of three-dimensional elasticity leads to the strains of the plate

$$\left(\text{for 3d, } \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\right)$$

Weak Form of Reissner-Mindlin Plate -1-

- Axial strains and in-plane shear: $\epsilon_{\alpha\eta} = \underbrace{-\frac{1}{2}(\beta_{\alpha,\eta} + \beta_{\eta,\alpha})}_{\kappa_{\alpha\eta}} x_3 = \kappa_{\alpha\eta} x_3$
- Out-of-plane shear: $\epsilon_{\alpha 3} = \frac{1}{2}(-\beta_\alpha + u_{3,\alpha}) = \frac{1}{2}\gamma_\alpha$
  - Note that always $\epsilon_{\alpha 3} = \epsilon_{3\alpha}$
- Through-the-thickness strain: $\epsilon_{33} = 0$

- The plate strains introduced into the internal virtual work of three-dimensional elasticity give the internal virtual work of the plate

$$\int_\Omega \int_{-t/2}^{t/2} [\sigma_{\alpha\eta}\epsilon_{\alpha\eta}(\phi_\gamma) + 2\sigma_{\alpha 3}\epsilon_{\alpha 3}(\phi_\gamma, v_3)] \, dx_3 d\Omega =$$

  - with virtual displacements and rotations: $\phi_\gamma, \, v_3$

$$= \int_\Omega \int_{-t/2}^{t/2} [\sigma_{\alpha\eta} x_3 \kappa_{\alpha\eta}(\phi_\gamma) + \sigma_{\alpha 3}\gamma_\alpha(\phi_\gamma, v_3)] \, dx_3 d\Omega = \int_\Omega m_{\alpha\eta}\kappa_{\alpha\eta}(\phi_\gamma) + s_\alpha \gamma_\alpha(\phi_\gamma, v_3) \, d\Omega$$

Weak Form of Reissner-Mindlin Plate -2-

- Definition of bending moments: $m_{\alpha\eta} = \int_{-t/2}^{t/2} \sigma_{\alpha\eta} x_3 \, dx_3$
- Definition of shear forces: $s_\alpha = \int_{-t/2}^{t/2} \sigma_{\alpha 3} \, dx_3$

External virtual work

- Distributed surface load

$$\int_\Omega q v_3 \, d\Omega$$

Weak form of Reissner-Mindlin plate

$$\int_\Omega \left[m_{\alpha\eta} \kappa_{\alpha\eta}(\phi_\gamma) + s_\alpha \gamma_\alpha(\phi_\gamma, v_3) \right] d\Omega = \int_\Omega q v_3 \, d\Omega + \text{boundary terms}$$

- As usual summation convention applies

Weak Form of Reissner-Mindlin Plate -3-

- Constitutive equations
  - For bending moments (same as Kirchhoff plate)

$$\begin{bmatrix} m_{11} \\ m_{22} \\ m_{12} \end{bmatrix} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{bmatrix} \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} \end{bmatrix}$$

  - For shear forces

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = kGt \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \quad \text{with} \quad G = \frac{E}{2(1+\nu)} \quad k = \frac{5}{6} : \text{ shear correction factor}$$

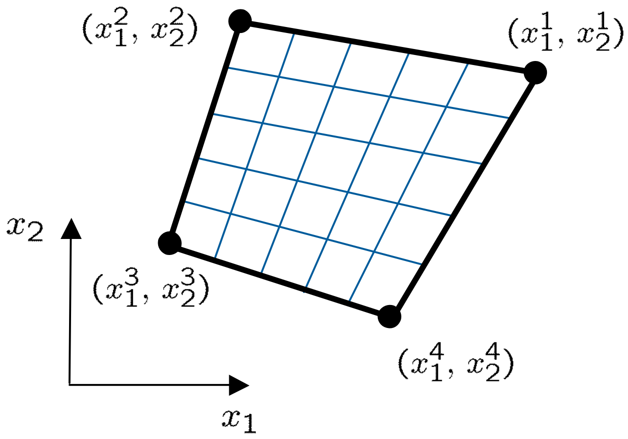
    - Note that the curvature $\kappa_{\alpha\beta}$ is a function of rotation angle β_α and the shear angle γ_α is a function of rotation angle β_α and the mid-surface displacement u_3

Finite Element Discretization -1-

- The independent variables in the weak form are β_α and u_3 and the corresponding test functions ϕ_α and v_3
  - Weak form contains only $\beta_\alpha,\ u_{3,\alpha}, \phi_\alpha$ and $v_{3,\alpha}$ so that C^0-interpolation is sufficient
    - Usual (Lagrange) shape functions such as used in 3D7 can be used

$$\beta_\alpha = \sum_{K=1}^{4} N^K \beta_\alpha^K \qquad u_3 = \sum_{K=1}^{4} N^K u_3^K$$

$$\phi_\alpha = \sum_{K=1}^{4} N^K \phi_\alpha^K \qquad v_3 = \sum_{K=1}^{4} N^K v_3^K$$



four-node isoparametric element

- Nodal values of variables: β_α^K and u_3^K
- Nodal values test functions: ϕ_α^K and v_3^K

- Interpolation equations introduced into the kinematic equations yield

$$\underbrace{\begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{bmatrix}}_{\boldsymbol{\kappa}} = - \sum_{K=1}^{4} \underbrace{\begin{bmatrix} 0 & N^K_{,x_1} & 0 \\ 0 & 0 & N^K_{,x_2} \\ 0 & N^K_{,x_1} & N^K_{,x_2} \end{bmatrix}}_{\boldsymbol{B}_b^K} \underbrace{\begin{bmatrix} u_3^K \\ \beta_1^K \\ \beta_2^K \end{bmatrix}}_{\boldsymbol{w}^K} \quad \Rightarrow \boldsymbol{\kappa} = - \sum_{K=1}^{4} \boldsymbol{B}_b^K \boldsymbol{w}^K$$

Finite Element Discretization -2-

$$\underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}}_{\gamma} = \sum_{K=1}^{4} \underbrace{\begin{bmatrix} N^K_{,x_1} & -N^K & 0 \\ N^K_{,x_2} & 0 & -N^K \end{bmatrix}}_{\boldsymbol{B}^K_s} \underbrace{\begin{bmatrix} u^K_3 \\ \beta^K_1 \\ \beta^K_2 \end{bmatrix}}_{\boldsymbol{w}^K}$$

$$\Rightarrow \gamma = \sum_{K=1}^{4} \boldsymbol{B}^K_s \boldsymbol{w}^K$$

- Constitutive equations in matrix notation
  - Bending moments: $\boldsymbol{m} = \boldsymbol{C}_b \boldsymbol{\kappa}$
  - Shear forces: $\boldsymbol{s} = \boldsymbol{C}_s \gamma$

- Element stiffness matrix of a four-node quadrilateral element
  - Bending stiffness (12x12 matrix)

$$\int_{\Omega_e} m_{\alpha\eta} \kappa_{\alpha\eta}(\phi_\gamma)\, d\Omega_e = \sum_{K=1}^{4} \sum_{M=1}^{4} \boldsymbol{w}^K \int_{\Omega_e} (\boldsymbol{B}^K_b)^T \boldsymbol{C}_b \boldsymbol{B}^M_b \, d\Omega_e \, \boldsymbol{v}^M$$

$$\Rightarrow \boldsymbol{k}_b = \sum_{K=1}^{4} \sum_{M=1}^{4} \int_{\Omega_e} \left(\boldsymbol{B}^K_b\right)^T \boldsymbol{C}_b \boldsymbol{B}^M_b \, d\Omega_e$$

Finite Element Discretization -3-

- Shear stiffness (12x12 matrix)

$$\int_{\Omega_e} s_{\alpha\eta}\gamma_\alpha(\phi_\gamma, v_3)\, d\Omega_e = \sum_{K=1}^{4}\sum_{M=1}^{4} \boldsymbol{w}^K \int_{\Omega_e} (\boldsymbol{B}_s^K)^T \boldsymbol{C}_s \boldsymbol{B}_s^M \, d\Omega_e \, \boldsymbol{v}^M$$

$$\Rightarrow \boldsymbol{k}_s = \sum_{K=1}^{4}\sum_{M=1}^{4} \int_{\Omega_e} \left(\boldsymbol{B}_s^K\right)^T \boldsymbol{C}_s \boldsymbol{B}_s^M d\Omega_e$$

- Element stiffness matrix

$$\boldsymbol{k} = \boldsymbol{k}_b + \boldsymbol{k}_s$$

  - The integrals are evaluated with numerical integration. If too few integration points are used, element stiffness matrix will be rank deficient.
    - The necessary number of integration points for the bilinear element are 2x2 Gauss points

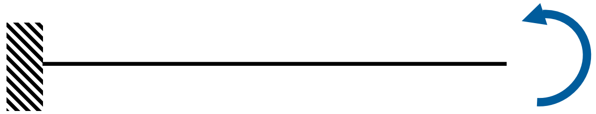
- The global stiffness matrix and global load vector are obtained by assembling the individual element stiffness matrices and load vectors
  - The assembly procedure is identical to usual finite elements

Shear Locking Problem -1-

- As discussed for the Bernoulli and Timoshenko beams with increasing plate slenderness physics dictates that shear deformations have to vanish (i.e., $\gamma_\alpha \rightarrow$ for $t \rightarrow 0$)

  - Reissner-Mindlin plate and Timoshenko beam finite elements have problems to approximate deformation states with zero shear deformations (shear locking problem)

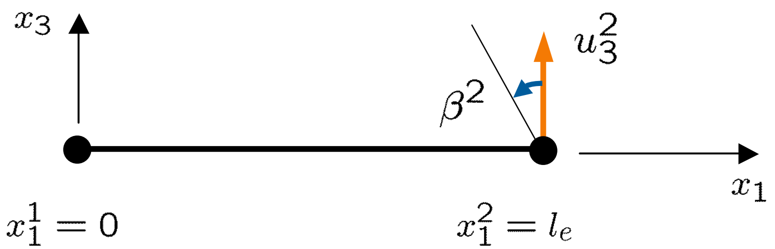
- 1D example: Cantilever beam with applied tip moment



  - Bending moment and curvature constant along the beam
  - Shear force and hence shear angle zero along the beam
  - Displacements quadratic along the beam

$$\underbrace{u_{3,\alpha}}_{\text{linear}} = \underbrace{\gamma_\alpha}_{=0} + \underbrace{\beta_\alpha}_{\text{linear}}$$

  - Discretized with one two-node Timoshenko beam element



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Shear Locking Problem -2-

- Deflection interpolation: $u_3 = \frac{x_1}{l_e} u_3^2$
- Rotation interpolation: $\beta = \frac{x_1}{l_e} \beta^2$
- Shear angle: $\gamma = \frac{u_3^2}{l_e} - \frac{x_1}{l_e} \beta^2$
- For the shear angle to be zero along the beam, the displacements and rotations have to be zero. Hence, a shear strain in the beam can only be reached when there are no deformations!
- Similarly, enforcing $\gamma = \frac{u_3^2}{l_e} - \frac{x_1}{l_e} \beta^2 = 0$ at two integration points leads to zero displacements and rotations!
- However, enforcing $\gamma = \frac{u_3^2}{l_e} - \frac{x_1}{l_e} \beta^2 = 0$ only at one integration point (midpoint of the beam) leads to non-zero displacements $u_3^2 = \frac{l_e}{2} \beta^2$

- In the following several techniques will be introduced to circumvent the shear locking problem
  - Use of higher-order elements
  - Uniform and selective reduced integration
  - Discrete Kirchhoff elements
  - Assumed strain elements

Constraint Ratio (Hughes et al.) -1-

- Constraint ratio is a semi-heuristic number for estimating an element's tendency to shear lock
  - Continuous problem
    - Number of equilibrium equations: 3 (two for bending moments + one for shear force)
    - Number of shear strain constraints in the thin limit: 2

$$\gamma_\alpha = u_{3,\alpha} - \beta_\alpha = 0 \quad \text{for } \alpha = 1, 2$$

    - Constraint ratio: $\frac{\text{"number of equations"}}{\text{"number of constraints"}} = \frac{3}{2} = 1.5$
  - With four-node quadrilateral finite elements discretized problem
    - Number of degrees of freedom per element on a very large mesh is ~3
    - Number of constraints per element for 2x2 integration per element is 8

$$\gamma_\alpha = u_{3,\alpha} - \beta_\alpha = 0 \quad \text{for } \alpha = 1, 2 \text{ at four intergation points}$$

      - Constraint ratio: $\frac{3}{8} = 0.375$ ↯
    - Number of constraints per element for one integration point per element is 2
      - Constraint ratio: $\frac{3}{2} = 1.5$ ✓

Constraint Ratio (Hughes et al.) -2-

- Constraint ratio for a 9 node element
  - Number of degrees of freedom per element on a very large mesh is ~ 4x3 =12
  - Number of constraints per element for 3x3 integration is 18
  - Constraint ratio: $\frac{12}{18} = 0.667$

- Constraint ratio for a 16 node element
  - Number of degrees of freedom per element on a very large mesh is ~ 9x3=27
  - Number of constraints per element for 4x4 integration is 32
  - Constraint ratio: $\frac{27}{32} = 0.844$

- As indicated by the constraint ratio higher-order elements are less likely to exhibit shear locking

Uniform And Selective Reduced Integration -1-

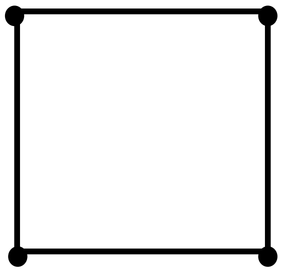
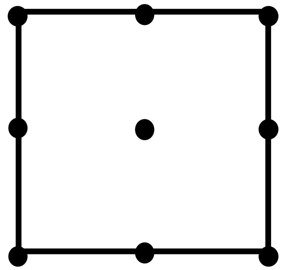
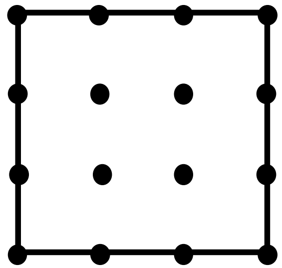
- The easiest approach to avoid shear “locking” in thin plates is to use some form of reduced integration

  - In uniform reduced integration the bending and shear terms are integrated with the same rule, which is lower than the “normal”

  - In selective reduced integration the bending term is integrated with the normal rule and the shear term with a lower-order rule

- Uniform reduced integrated elements have usually rank deficiency (i.e. there are internal mechanisms; deformations which do not need energy)
  - The global stiffness matrix is not invertible
  - Not useful for practical applications

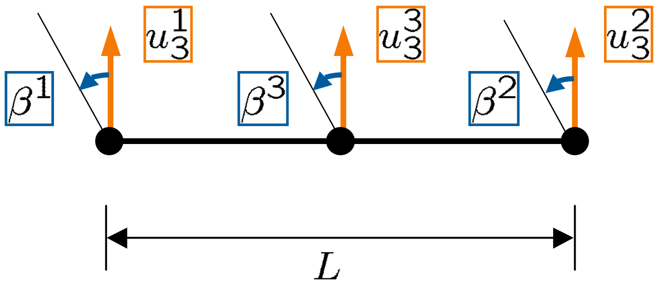
Uniform And Selective Reduced Integration -2-

			
Shape functions	Bilinear	Biquadratic	Bicubic
Uniform reduced integration	1x1	2x2	3x3
Selective reduced integration	1x1 shear 2x2 bending	2x2 shear 3x3 bending	3x3 shear 4x4 bending

- “Shear” refers to the integration of the element shear stiffness matrix $\boldsymbol{k}_s$
- “Bending” refers to the integration of the element bending stiffness matrix $\boldsymbol{k}_b$

Discrete Kirchhoff Elements

- The principal approach is best illustrated with a Timoshenko beam
- The displacements and rotations are approximated with quadratic shape functions



$$u_3(x) = u_3^3 + (u_3^2 - u_3^1)\frac{x}{L} + 2(u_3^1 + u_3^2 - 2u_3^3)\left(\frac{x}{L}\right)^2$$

$$\beta(x) = \beta^3 + (\beta^2 - \beta^1)\frac{x}{L} + 2(\beta^1 + \beta^2 - 2\beta^3)\left(\frac{x}{L}\right)^2$$

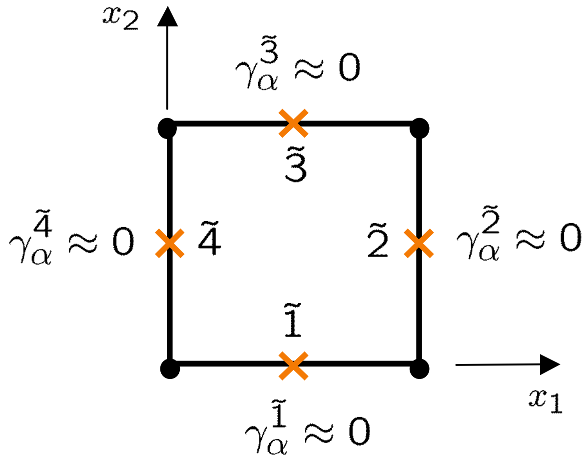
- The inner variables are eliminated by enforcing zero shear stress at the two gauss points $\pm\frac{1}{\sqrt{3}}\frac{L}{2}$

$$u_3^3 = \frac{1}{2}(u_3^1 + u_3^2) - \frac{1}{8}L(\beta^1 - \beta^2) \qquad \beta^3 = \frac{3}{2L}(u_3^1 - u_3^2) - \frac{1}{4}(\beta^1 + \beta^2)$$

- Back inserting u_3^3 and β^3 into the interpolation equations leads to a beam element with 4 nodal parameters

Assumed Strain Elements

- It is assumed that the out-of-plane shear strains at edge centres are of "higher quality" (similar to the midpoint of a beam)



  - First, the shear angle at the edge centres $(\xi_{\tilde{I}}, \eta_{\tilde{I}})$ is computed using the displacement and rotation nodal values

$$\gamma_\alpha^{\tilde{I}} = \sum_{K=1}^{4} \boldsymbol{B}_s^K(\xi_{\tilde{I}}, \eta_{\tilde{I}})\boldsymbol{w}^K$$

  - Subsequently, the shear angles from the edge centres are interpolated back

$$\gamma_\alpha = \sum_{\tilde{I}=1}^{4} \tilde{N}^{\tilde{I}} \gamma_\alpha^{\tilde{I}} = \sum_{\tilde{I}=1}^{4} \tilde{N}^{\tilde{I}} \sum_{K=1}^{4} \boldsymbol{B}_s^K(\xi_{\tilde{I}}, \eta_{\tilde{I}})\boldsymbol{w}^K$$

    - Note that the shear functions $\tilde{N}^{\tilde{I}}$ are special edge shape functions

  - These reinterpolated shear angles are introduced into the weak form and are for element stiffness matrix computation used