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Symmetric prismatic tensegrity structures: Part I. Configuration and stability

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ABSTRACT

This paper presents a simple and efficient method to determine the self-equilibrated configurations of prismatic tensegrity structures, nodes and members of which have dihedral symmetry. It is demonstrated that stability of this class of structures is not only directly related to the connectivity of members, but is also sensitive to their geometry (height/radius ratio), and is also dependent on the level of self-stress and stiffness of members. A catalogue of the structures with relatively small number of members is presented based on the stability investigations.

1. Introduction

In this paper, we describe a study into the configuration and stability of prismatic tensegrity structures with dihedral symmetry. The simplest example of this class of structures is shown in Fig. 1. This class of structures was studied by Connelly and Terrell (1995): they showed that the example shown in Fig. 1, and other prismatic tensegrity structures where the horizontal cables are connected to adjacent nodes, are *super stable*. Super stable structures are guaranteed to be stable, for any level of self-stress and material properties, as long as every member has a positive rest-length (by contrast, see Schenk et al., 2007 for an example of a prismatic tensegrity structure where some of the members have *zero* rest length). Connelly and Terrell (1995), however, did not address the stability of other prismatic tensegrity structures that are not super stable. These structures may still be stable under certain conditions; investigation of these conditions and classification of the stability of prismatic tensegrity structures are the subjects of this paper.

We will use three different meanings of stability in this paper. We describe a structure as *stable* when it has a positive definite tangent stiffness matrix: in general, the tangent stiffness matrix depends upon both the material properties of the members, and the level of self-stress (Guest, 2006) (note that we are not considering any concept of higher-order stability, as discussed in Connelly and Servatius, 1994). We describe a structure as *prestress stable* when an idealized version of the structure, with members whose length cannot change, is stable. Prestress stability depends upon the geometric configuration of the structure; a 'stretched' version of a prestress stable structure may not be prestress stable. (By 'stretched' we do not mean the tensegrity itself has been deformed, rather we mean a new tensegrity where the coordinates of the nodes are given by an affine transformation of the coordinates of the original nodes.) Finally we use the term *super stable* in the same sense as that used by Connelly and Terrell (1995): it implies that any stretched configuration of the structure is stable.

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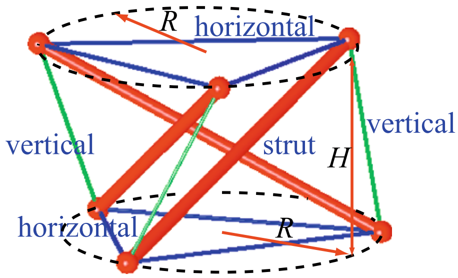


Fig. 1. The simplest prismatic tensegrity structure in three-dimensional space. The thin and thick lines denote, respectively, cables that carry tension, and struts that carry compression. The nodes lie in two horizontal planes. This structure has $\mathbf{D}_3$ symmetry, and using the notation described at the end of Section 2.1, is denoted $\mathbf{D}_3^{1,1}$.

After showing that some prismatic tensegrity structures are super stable, Connelly and Terrell (1995) listed the following three questions, where the terms 'rigid' and 'tensigrid' denote prestress stable and tensegrity structure, respectively:

(1) Can other methods be applied to show that some of the other prismatic tensigrids are rigid?
(2) Can it be shown that some of the other prismatic tensigrids are not rigid?
(3) How "often" it is rigid?

In this study, we show that stability of prismatic tensegrity structures is dependent on the connectivity of the members (horizontal cables and vertical cables), the height/radius ratio, and the self-stress to member stiffness ratio. It is shown that structures that are not super stable can still be stable in some cases. For example, the structure shown in Fig. 2(a) is not super stable, but it can be prestress stable if it has the right height/radius ratio.

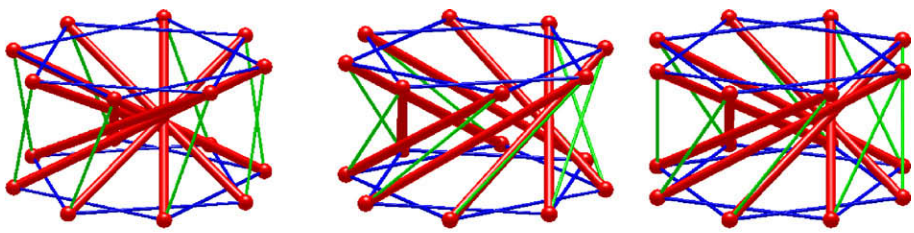
Following this introduction, the study is organized as follows. Section 2 presents a simple method for the determination of self-equilibrated configurations of prismatic tensegrity structures making use of their symmetry. Conditions for the divisible structures, which can be physically divided into several identical substructures, are given in Section 3. Section 4 defines stability criteria, and discusses the critical parameters for the stability of prismatic tensegrity structures. Section 5 gives a catalogue of the stability of prismatic tensegrity structures with up to 10 struts, and Section 6 concludes the paper.

2. Symmetry and configuration

We define the class of prismatic tensegrity structures as follows. The structures have $2n$ nodes, arranged in two horizontal circles of radius R around the vertical z-axis, which is an n-fold symmetry axis. Within each circle, each node is connected by 'horizontal' cables to two other nodes. The two planes containing the nodes are at $z = \pm H/2$. Each node is connected by a strut and a 'vertical' cable to nodes in the other plane. The structure has $\mathbf{D}_n$ symmetry, using the Schoenflies notation, and this symmetry allows us to calculate self-equilibrated configurations by considering the equilibrium equations of only one node.

2.1. Orbits

Consider a specific set of elements (nodes or members) of a structure with symmetry G. If one element in a set can be transformed to any other member of that set by a proper symmetry operation in G, then this set of elements are said to belong to the same *orbit*. A structure can have several different orbits of elements of the same type.



(a) prestress stable ($\mathbf{D}_8^{2,3}$) (b) not stable ($\mathbf{D}_8^{2,1}$) (c) divisible ($\mathbf{D}_8^{2,2}$)

Fig. 2. Prismatic tensegrity structures with $\mathbf{D}_8$ symmetry. The structure $\mathbf{D}_8^{2,3}$ is prestress stable when its height/radius ratio is within the range of [0.4, 3.1]; the structure $\mathbf{D}_8^{2,1}$ can never be stable, and the structure $\mathbf{D}_8^{2,2}$ can be physically divided into two identical substructures $\mathbf{D}_4^{1,1}$.

We are considering structures that have dihedral symmetry, denoted $\mathbf{D}_n$: there is a single major n-fold rotation (C_n^i) axis, which we assume is the vertical, z-axis, and n 2-fold rotation (C_{2j}) axes perpendicular to this axis (Kettle, 1995). In total there are $2n$ symmetry operations.

For a prismatic tensegrity structure, there is one orbit of nodes, and each symmetry operation transforms a reference node into one of the other nodes; there is a one-to-one correspondence between the nodes and the symmetry operations. (When there is a one-to-one correspondence between elements and symmetry operations, the orbit is called a *regular orbit*). There are in total $2n$ nodes, arranged in two horizontal planes, with n nodes in each. We number the nodes from 0 to $n-1$ in the top plane, and n to $2n-1$ in the bottom plane. An example structure with $\mathbf{D}_3$ symmetry is shown in Fig. 3: nodes N_0, N_1, N_2, and nodes N_3, N_4, N_5 lie in the top and bottom horizontal planes, respectively. Any node, e.g., node N_0, can be transformed to any other node, including itself, by one of the symmetry operations of $\mathbf{D}_3$ as listed in Table 1.

There are three orbits of members: horizontal cables, vertical cables, and struts. Each node is connected by two horizontal cables lying in a horizontal plane, one vertical cable, and one strut: the vertical cable and strut connect nodes in different planes. The members in each orbit have the same length; we assume a symmetric internal self-stress state, and hence the internal force, and the force density (internal force to length ratio) are also the same in each member of an orbit. There are $2n$ horizontal cables, and each symmetry operation transforms a reference cable into one of the other cables; there is a one-to-one correspondence between the horizontal cables and the symmetry operations (the horizontal cables form a regular orbit). There are, however, only n vertical cables, and n struts; there is a one-to-two correspondence between the vertical cables (or struts) and the symmetry operations. Each vertical cable and strut intersects one of the 2-fold horizontal rotation axes, and this 2-fold operation transforms the vertical cable (or strut) into itself. For example, transformations of the members of the structure with $\mathbf{D}_3$ symmetry by the symmetry operations are listed in Table 1. For some structures, the horizontal cables may cross one another; we neglect to consider any interference, essentially assuming that these cables can pass through one another.

We use the notation $\mathbf{D}_n^{h,v}$ to describe the connectivity of a prismatic tensegrity with $\mathbf{D}_n$ symmetry: h and v, respectively, describe the connectivity of the horizontal and vertical cables, while that of struts is fixed. We describe the connectivity of a reference node N_0 as follows – all other connections are then defined by the symmetry.

(1) Without loss of generality, we assume that a strut connects node N_0 in the top plane to node N_n in the bottom plane.
(2) A horizontal cable connects node N_0 to node N_h: symmetry also implies that a horizontal cable must also connect node N_0 to node N_{n-h}. We restrict $1 \leqslant h \leqslant n/2$.
(3) A vertical cable connects node N_0 in the top plane to node N_{n+v} in the bottom plane. We restrict $1 \leqslant v \leqslant n/2$ (choosing $n/2 \leqslant v \leqslant n$ would give essentially the same set of structures, but in left-handed versions).

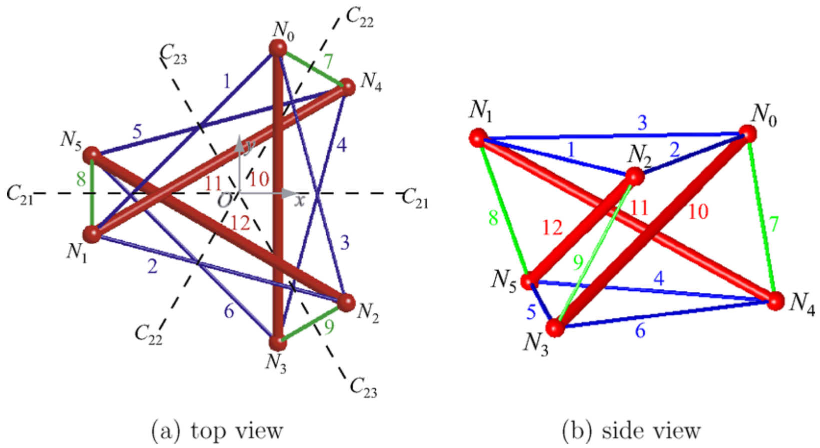


Fig. 3. The prismatic tensegrity structure $\mathbf{D}_3^{1,1}$.

Table 1
Transformation of nodes and members of the structure $\mathbf{D}_3^{1,1}$ in Fig. 3 corresponding to the symmetry operations of $\mathbf{D}_3$

	$E(C_3^0)$	C_3^1	C_3^2	C_{21}	C_{22}	C_{23}	
Node N_0	N_0	N_1	N_2	N_3	N_4	N_5	Nodes
Member 1	1	2	3	6	4	5	Horizontal cables
Member 7	7	8	9	9	7	8	Vertical cables
Member 10	10	11	12	10	11	12	Struts

The elements listed in the left-hand column are transformed to the elements shown in the table by the symmetry operations given in the top row.

2.2. Transformation matrices

Let $\mathbf{x}_0$ and $\mathbf{x}_i$ ($\in \Re^3$) denote the coordinates of nodes N_0 and N_i in three-dimensional space, respectively. Suppose that node N_0 can be transformed to node N_i by a symmetry operation in the group $\mathbf{D}_n$. Then we have the following equation with the *transformation matrix* $\mathbf{R}_i \in \Re^{3\times3}$:

$$\mathbf{x}_i = \mathbf{R}_i\mathbf{x}_0 \quad \text{for } i = 1, 2, \ldots, 2n. \tag{1}$$

Because the nodes form a regular orbit, there will be one matrix $\mathbf{R}_i$ for each symmetry operation in the group. These matrices are said to form a *representation* Γ_{xyz} of the group $\mathbf{D}_n$. To make use of this representation, we will use some *group representation theory*; an introduction to this material can be found, for example, in Bishop (1973).

The matrices $\mathbf{R}_i$ form a *reducible* representation of $\mathbf{D}_n$. However, it is straightforward to write this reducible representation in terms of *irreducible representations*. The irreducible representations that make up Γ_{xyz} can be read off from a set of character tables, e.g., Altmann and Herzig (1994). For any $\mathbf{D}_n$, Γ_{xyz} is the direct sum of the irreducible representations A_2 and E_1 (the standard notation is E for $\mathbf{D}_3$ and $\mathbf{D}_4$, but we will use E_1 for these cases too). The irreducible representation A_2 is one-dimensional, and corresponds to the transformation of the z-coordinate. The irreducible representation E_1 is two-dimensional and corresponds to the transformation of the x- and y-coordinates. Thus, the transformation matrices $\mathbf{R}_i \in \Re^{3\times3}$ can be written as

$$\mathbf{R}_i = \begin{bmatrix} \mathbf{R}_i^{E_1} & \\ & \mathbf{R}_i^{A_2} \end{bmatrix}, \tag{2}$$

where the matrices $\mathbf{R}_i^{E_1} \in \Re^{2\times2}$ form the representation E_1 and the matrices $\mathbf{R}_i^{A_2} \in \Re^{1\times1}$ form the representation A_2.

The one-dimensional matrices $\mathbf{R}_i^{A_2}$ are unique, but there is some limited choices for the two-dimensional matrices $\mathbf{R}_i^{E_1}$. By choosing a positive rotation around the z-axis for $\mathbf{R}_1$, the transformation matrix $\mathbf{R}_i$ for the cyclic rotation C_n^i through $2i\pi/n$ can be written as

$$\mathbf{R}_i = \begin{bmatrix} C_i & -S_i & 0 \\ S_i & C_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for } 0 \leqslant i \leqslant n-1, \tag{3}$$

where $C_i = \cos(2i\pi/n)$ and $S_i = \sin(2i\pi/n)$, and i is running from 0 to $n-1$. By choosing that a dihedral rotation about the x-axis transforms node N_0 to node N_n, the transformation matrices $\mathbf{R}_i$ for the 2-fold rotations can be written as

$$\mathbf{R}_i = \begin{bmatrix} C_i & S_i & 0 \\ S_i & -C_i & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{for } n \leqslant i \leqslant 2n-1. \tag{4}$$

2.3. Symmetric state of self-stress

There is only one orbit of nodes, and hence to find a totally symmetric state of self-stress, we only need to consider equilibrium of one node under zero external loading: equilibrium of any other node is identical, by symmetry (Connelly and Back, 1998).

Consider a single reference node N_0, and the members that are connected to it – an example is shown in Fig. 4. The coordinates $\mathbf{x}_h$ and $\mathbf{x}_{n-h}$ of the nodes N_h and N_{n-h} connected to the reference node as horizontal cables can be computed as follows by using Eq. (1):

$$\begin{aligned} \mathbf{x}_h &= \mathbf{R}_h\mathbf{x}_0, \\ \mathbf{x}_{n-h} &= \mathbf{R}_{n-h}\mathbf{x}_0, \end{aligned} \tag{5}$$

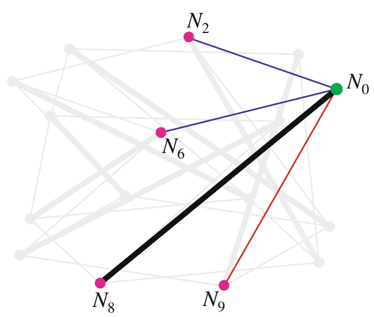


Fig. 4. All nodes connected to a reference node N_0 of the structure $\mathbf{D}_8^{2,1}$.

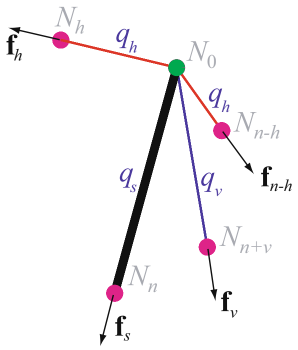


Fig. 5. Self-equilibrium of the reference node of prismatic tensegrity structures. The three cable forces, $\mathbf{f}_h$, $\mathbf{f}_{n-h}$ and $\mathbf{f}_v$ are all tensile, and have a positive magnitude; the strut force $\mathbf{f}_s$ is compressive, and has a negative magnitude.

and the direction vectors $\mathbf{d}_h$ and $\mathbf{d}_{n-h}$ of the horizontal cables can be written as

$$\begin{aligned} \mathbf{d}_h &= \mathbf{x}_h - \mathbf{x}_0 = (\mathbf{R}_h - \mathbf{I}_{3\times3})\mathbf{x}_0, \\ \mathbf{d}_{n-h} &= \mathbf{x}_{n-h} - \mathbf{x}_0 = (\mathbf{R}_{n-h} - \mathbf{I}_{3\times3})\mathbf{x}_0, \end{aligned} \tag{6}$$

where $\mathbf{I}_{3\times3}$ denotes the 3-by-3 identity matrix. Similarly, the coordinates $\mathbf{x}_s$ and $\mathbf{x}_v$ of the nodes N_n and N_{n+v} in the bottom plane that are connected to N_0 by a strut and a vertical cable, respectively, can be calculated by

$$\begin{aligned} \mathbf{x}_s &= \mathbf{R}_n\mathbf{x}_0, \\ \mathbf{x}_v &= \mathbf{R}_{n+v}\mathbf{x}_0, \end{aligned} \tag{7}$$

and their direction vectors $\mathbf{d}_s$ and $\mathbf{d}_v$ are

$$\begin{aligned} \mathbf{d}_s &= \mathbf{x}_s - \mathbf{x}_0 = (\mathbf{R}_n - \mathbf{I}_{3\times3})\mathbf{x}_0, \\ \mathbf{d}_v &= \mathbf{x}_v - \mathbf{x}_0 = (\mathbf{R}_{n+v} - \mathbf{I}_{3\times3})\mathbf{x}_0. \end{aligned} \tag{8}$$

Let q_h, q_s and q_v denote the force densities of the horizontal cables, strut and vertical cable, respectively, where the force density is the ratio of the axial force f_i to the length l_i; i.e., $q_i = f_i/l_i$. Because tensegrity structures are pin-jointed and carry only axial forces in the members, the direction of the axial force is identical to that of the member. Thus, the axial force vectors $\mathbf{f}_h$ and $\mathbf{f}_{n-h}$ of the horizontal cables can be written as (Fig. 5)

$$\begin{aligned} \mathbf{f}_h &= f_h\mathbf{d}_h/l_h = q_h\mathbf{d}_h = q_h(\mathbf{R}_h - \mathbf{I}_{3\times3})\mathbf{x}_0, \\ \mathbf{f}_{n-h} &= f_h\mathbf{d}_{n-h}/l_h = q_h\mathbf{d}_{n-h} = q_h(\mathbf{R}_{n-h} - \mathbf{I}_{3\times3})\mathbf{x}_0. \end{aligned} \tag{9}$$

Similarly, the axial force vectors $\mathbf{f}_s$ and $\mathbf{f}_v$ of the strut and vertical cable are

$$\begin{aligned} \mathbf{f}_s &= q_s(\mathbf{R}_{n+s} - \mathbf{I}_{3\times3})\mathbf{x}_0, \\ \mathbf{f}_v &= q_v(\mathbf{R}_{n+v} - \mathbf{I}_{3\times3})\mathbf{x}_0. \end{aligned} \tag{10}$$

When no external load is applied, the node N_0 should be in equilibrium, i.e.,

$$\mathbf{f}_h + \mathbf{f}_{n-h} + \mathbf{f}_s + \mathbf{f}_v = \mathbf{0}. \tag{11}$$

Substituting Eqs. (9) and (10) into Eq. (11), it gives

$$\widetilde{\mathbf{S}}_{xyz}\mathbf{x}_0 = \mathbf{0}, \tag{12}$$

where

$$\widetilde{\mathbf{S}}_{xyz} = 2q_h\begin{bmatrix} C_h - 1 & 0 & 0 \\ 0 & C_h - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + q_s\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} + q_v\begin{bmatrix} C_v - 1 & S_v & 0 \\ S_v & -C_v - 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{13}$$

$\widetilde{\mathbf{S}}_{xyz}$ is a block-diagonal matrix constructed from a 2×2 and a 1×1 sub-matrices on its leading diagonal. Both of these sub-matrices should be singular to allow the solution of Eq. (12) to give the position vector $\mathbf{x}_0$ of the reference node with non-trivial coordinates in three-dimensional space. For the singularity of the 1×1 sub-matrix, we have

$$0 - 2q_s - 2q_v = 0, \tag{14}$$

i.e.,

$$q_v = -q_s. \tag{15}$$

For the 2×2 sub-matrix, we can enforce singularity by ensuring that the determinant is equal to zero, i.e.,

$$[2q_h(C_h - 1) + 0 + q_v(C_v - 1)][2q_h(C_h - 1) - 2q_s - q_v(C_v + 1)] - q_v^2 S_v^2 = 0. \tag{16}$$

Using $q_v = -q_s$ from Eq. (15), and the trigonometric relationship $C_v^2 + S_v^2 = 1$, Eq. (16) reduces to

$$4\left(\frac{q_h}{q_v}\right)^2 (C_h - 1)^2 + 2C_v - 2 = 0. \tag{17}$$

Since both of q_h and q_v should have positive sign (they are both cables in tension), only the positive solution is adopted, i.e.,

$$\frac{q_h}{q_v} = +\frac{\sqrt{2 - 2C_v}}{2(1 - C_h)}. \tag{18}$$

When both Eqs. (15) and (18) hold, $\widetilde{\mathbf{S}}$ has a nullity of 2, and hence has a two-dimensional null-space. Any vector in that null-space can be the coordinate vector $\mathbf{x}_0$ of the reference node. In general, the coordinate vector can be written in terms of two parameters, R and H, as

$$\mathbf{x}_0 = \frac{R}{R_0}\begin{bmatrix} C_v - 1 + \sqrt{2 - 2C_v} \\ S_v \\ 0 \end{bmatrix} + \frac{H}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{19}$$

where R_0 is the norm of the first vector representing the coordinates in xy-plane, and then R and H denote the radius and height of the structure, which can have arbitrary real values. Connectivity of horizontal cables does not affect the self-equilibrated configuration of prismatic tensegrity structures, but, as we will see in Section 4, it affects the stability of the structures.

By the application of Eqs. (1), (3) and (4), the coordinates of all the other nodes N_i can be determined by running i from 1 to $2n - 1$.

3. Divisibility conditions

Depending on the connectivity of members, a prismatic tensegrity structure may be completely separated into several identical substructures that have no mechanical relation with each other. The substructures are of lower symmetry compared to the original structure. For example, the structure $\mathbf{D}_6^{2,2}$ in Fig. 6(a) can be divided into two identical substructures $\mathbf{D}_3^{1,1}$. We will exclude divisible structures from our stability investigation, because there is nothing to prevent the substructures moving relative to one another; the stability of the substructures themselves will be considered anyway for the lower symmetry case.

This section presents the necessary and sufficient divisibility conditions for prismatic tensegrity structures. It is demonstrated that divisibility of these structures depends on the connectivity of the horizontal as well as vertical cables. The divisibility conditions for prismatic tensegrity structures have also been studied by Hinrichs (1984), using star polygons that have been established, for example, in Coxeter (1973). The indivisible circuit of horizontal cables in this study is referred to as a real or proper star polygon that has a single connected network, and the divisible circuit is referred to as a 'compound star polygon', in Coxeter (1973).

3.1. Divisibility of horizontal cables

Suppose that we randomly select one node as the starting node, and travel to the next along the horizontal cables in the same horizontal plane. If we repeat this in a consistent direction, eventually, we must come back to the starting node.

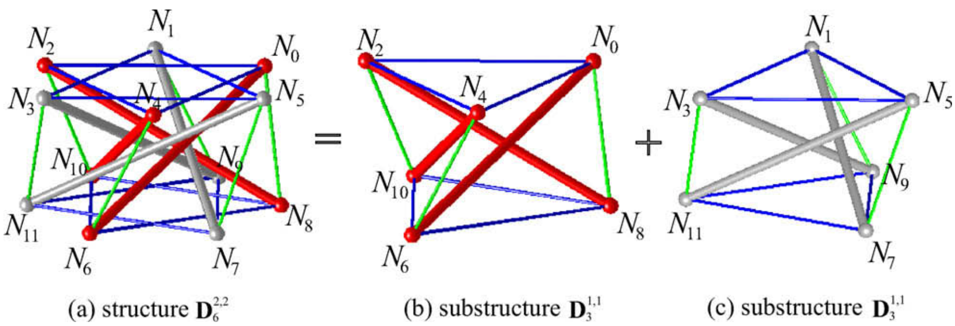


Fig. 6. Divisible structure $\mathbf{D}_6^{2,2}$ and its substructures $\mathbf{D}_3^{1,1}$.

The nodes and horizontal cables that have been visited in the trip are said to belong to the same *circuit*. If there are more than one circuits in the plane, the horizontal cables are said to be *divisible*; otherwise, they are *indivisible*.

Denote the number of circuits of the horizontal cables in one plane by n^c, and the number of nodes in a circuit by n^s. Each time we travel along a horizontal cable of the circuit, we pass by h nodes, and hence by the time we return to the starting node, we have passed hn^s nodes. Suppose that, in this circuit, we have travelled around the plane h^s times, and have hence passed nh^s nodes. Thus

$$n^s h = nh^s. \tag{20}$$

The number of circuits n^c in each horizontal plane is then given by

$$n^c = \frac{n}{n^s} = \frac{h}{h^s}. \tag{21}$$

The necessary and sufficient condition for the divisibility of horizontal cables in the same plane is that there is more than one circuit of nodes; i.e., $n^c \neq 1$. And hence, we have

$$h \neq h^s. \tag{22}$$

If the structure is divisible, the above parameters give useful information about the substructures. There will be n^c substructures, and they will have n^s nodes in each plane, with a connectivity of the horizontal cables of h^s.

Consider, for example, the divisible structure $\mathbf{D}_6^{2,2}$ shown in Fig. 6(a): node N_0 is connected to nodes N_2 and N_4 by the horizontal cables in the upper plane. It is easy to see that these three nodes form a circuit. This circuit does not have any mechanical relation with the other constituted by the nodes N_1, N_3 and N_5. The same situation occurs for the horizontal cables in the bottom plane. Therefore, the structure has in total four circuits, two in each plane:

Circuit	Nodes
1	N_0, N_2, N_4
2	N_1, N_3, N_5
3	N_6, N_8, N_{10}
4	N_7, N_9, N_{11}

(23)

In this case, travelling along one circuit takes us around the z-axis only once, but this is not always the case. For example, consider one of the planes of the structure with $\mathbf{D}_{14}$ symmetry as shown in Fig. 7; we can have the following cases where the horizontal cables are divisible:

(1) In the case of $h = 2$, as shown in Fig. 7(a), the horizontal cables in the plane can be divided into two circuits ($n^c = 2$), seven nodes in each ($n^s = 7$). The horizontal cables connect each node to the adjacent node in the circuit ($h^s = 1$).
(2) When $h = 4$, as shown in Fig. 7(b), the horizontal cables are divisible, with seven nodes in each circuit. For each circuit, the horizontal cables now connect a node to the second node away in that circuit, i.e., $h^s = 2$.
(3) When $h = 6$, as shown in Fig. 7(c), the horizontal cables are again divisible. Now for each circuit, the horizontal cables connect a node to the third node away in that circuit, i.e., $h^s = 3$.

Note that Eq. (22) is only the divisibility condition for the horizontal cables but not for the whole structure. For example, the structure $\mathbf{D}_6^{2,1}$ in Fig. 8(a) has two circuits of horizontal cables in each plane of nodes. However, those circuits are all connected by the struts and vertical cables, and the structure is indivisible. Hence, connectivity of vertical cables, which connect the circuits in different horizontal planes, should also be taken into consideration.

3.2. Divisibility of vertical cables

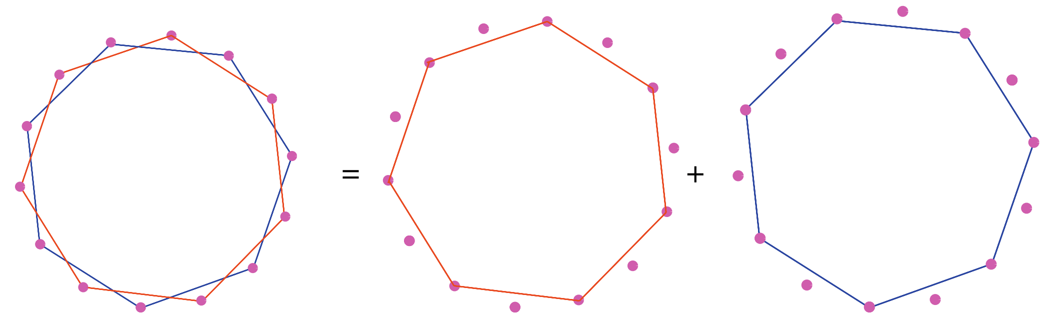
Suppose that the horizontal cables are divisible: the nodes in the circuits of horizontal cables containing N_0 and N_n are

$$\begin{aligned} &\text{Circuit 1}: \quad N_0, N_h, N_{2h}, \ldots, N_{(n^s-1)h}, \\ &\text{Circuit 2}: \quad N_n, N_{n+h}, \ldots, N_{n+(n^s-1)h}. \end{aligned} \tag{24}$$

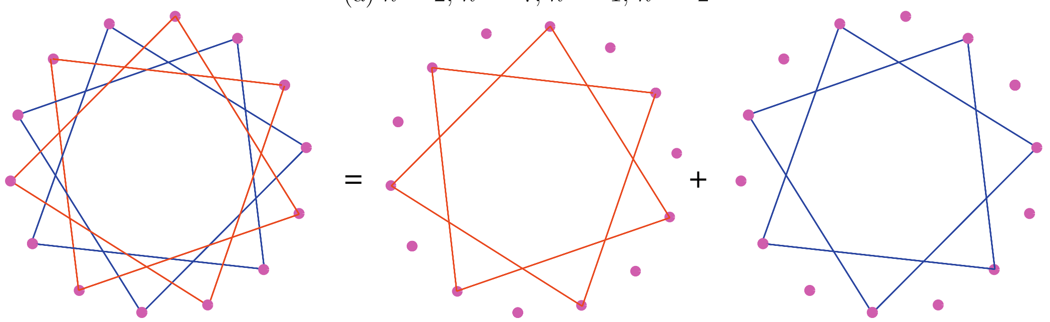
Circuit 1 and Circuit 2 are connected by struts from our assumption for the connectivity of struts. If they are also connected by vertical cables, then the substructure constructed from these nodes can be completely separated from the original structure. Thus, the structure is divisible if the horizontal cables are divisible, and the following relationship holds:

$$v = v^s h \quad \text{with } v^s \text{ integer}. \tag{25}$$

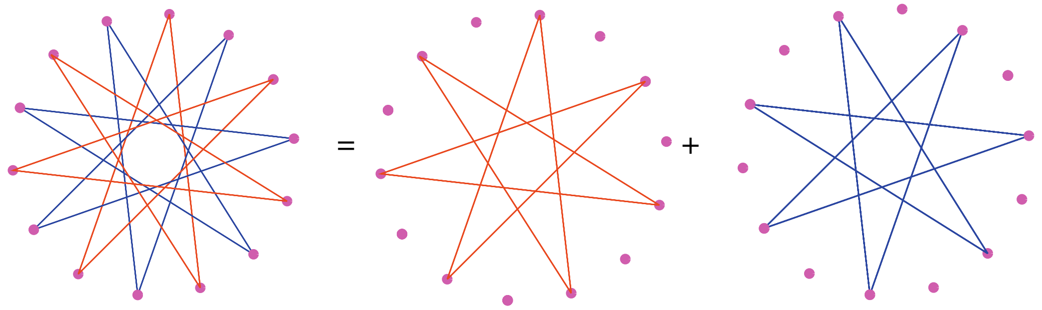
As contrasting examples, consider $\mathbf{D}_6^{2,2}$ and $\mathbf{D}_6^{2,1}$, which both have the same arrangement of (divisible) horizontal cables. The structure $\mathbf{D}_6^{2,2}$ shown in Fig. 6(a) satisfies Eq. (25) with $v^s = 1$ and hence is divisible. By contrast, the structure $\mathbf{D}_6^{2,1}$ in Fig. 8(a) has $v/h = 0.5$, does not satisfy Eq. (25), and is indivisible.



(a) $h = 2,\ n^s = 7,\ h^s = 1,\ n^c = 2$

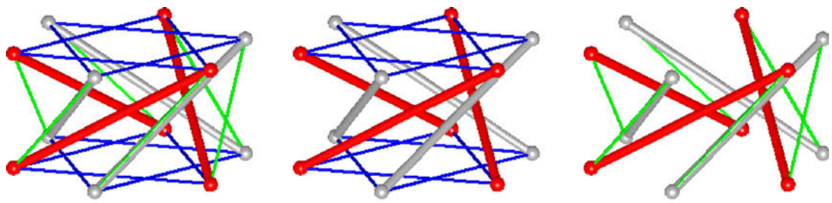


(b) $h = 4,\ n^s = 7,\ h^s = 2,\ n^c = 2$



(c) $h = 6,\ n^s = 7,\ h^s = 3,\ n^c = 2$

Fig. 7. An example of divisible horizontal cables ($n = 14$).



(a) indivisible (b) divisible (c) indivisible

Fig. 8. An example of indivisible structure, $\mathbf{D}_6^{2,1}$: (a) shows the entire structure; in (b), the vertical cables have been removed in the figure, to show that the remaining structure is divisible; in (c) the horizontal cables have been removed, showing that the vertical cables and the struts together connect all of the nodes, and the entire structure is therefore indivisible.

In summary, Eqs. (22) and (25) are the necessary and sufficient conditions for a divisible prismatic tensegrity structure. If both are satisfied, the original structure $\mathbf{D}_n^{h,v}$ can be divided into n^c identical substructures $\mathbf{D}_{n^s}^{h^s,v^s}$.

4. Stability

In this section, we will consider the stability of prismatic tensegrity structures. In particular, we will investigate the effect of a number of critical factors: the main one is the connectivity of the structure, but the height/radius ratio and the ratio of the stiffness to the force density of the members may also be important. All of the results were calculated using symmetry-adapted coordinates, and the common notation used in applied group representation theory is used to describe the results.

Although we will define here the stability criteria that we use, we will not derive all of the relevant matrices; instead we will use the formulation in Guest (2006) directly.

4.1. Stability criteria

From Guest (2006), the tangent stiffness matrix for a structure, $\mathbf{K}$ can be written as

$$\mathbf{K} = \mathbf{A}\widehat{\mathbf{G}}\mathbf{A}^{\mathrm{T}} + \mathbf{S}, \tag{26}$$

where $\mathbf{A}$ is the equilibrium matrix, which describes the geometry of the structure, $\mathbf{S}$ is the geometrical stiffness matrix (or called stress matrix in Guest (2006)), which depends on the connectivity of the structure and the level of stress, and $\widehat{\mathbf{G}}$ is a diagonal matrix containing modified axial stiffness for the members.

Rigid-body motions have no stiffness. To simplify the definitions, we will assume that rigid-body motions of the entire structure have been condensed out of the formulation, and the matrices are only related to displacements that cause deformation of the structure. Alternatively, the six zero eigenvalues of the stiffness matrices that correspond to rigid-body motions can be ignored in the stability investigation.

4.1.1. Stability

We say that a structure is *stable* when the tangent stiffness matrix $\mathbf{K}$ is positive definite. One way of considering positive-definiteness is to look at the eigenvalues of $\mathbf{K}$. The smallest eigenvalue is the minimum stiffness for some deformation of the structure; if the smallest eigenvalue of $\mathbf{K}$ is positive, then the structure is stable.

4.1.2. Prestress stability

For prestress stability, we consider the case when the members are axially rigid. In the formulation of Eq. (26), the diagonal entries in $\widehat{\mathbf{G}}$ become infinite. The only non-infinite eigenvalues of $\mathbf{K}$ will then come about from deformations that are in the nullspace of $\mathbf{A}^{\mathrm{T}}$ – first-order mechanisms of the structure. Consider that there are m mechanisms, and the mechanisms can be described by a set of basis vectors, $\mathbf{m}_1 \ldots \mathbf{m}_m$. If these mechanisms are written as the columns of a matrix $\mathbf{M}$:

$$\mathbf{M} = [\,\mathbf{m}_1 \quad \mathbf{m}_2 \quad \cdots \quad \mathbf{m}_m\,], \tag{27}$$

then a reduced stiffness matrix $\mathbf{Q}$ can be written as

$$\mathbf{Q} = \mathbf{M}^{\mathrm{T}}\mathbf{S}\mathbf{M}. \tag{28}$$

We say that a structure is *prestress stable* when the reduced stiffness matrix $\mathbf{Q}$ is positive definite; i.e., if the smallest eigenvalue of $\mathbf{Q}$ is positive, then the structure is prestress stable.

4.1.3. Super stability

Super stability guarantees that the reduced stiffness matrix is positive definite for any geometric realisation, i.e. for any R and H. We use the sufficient conditions for super stability of tensegrity structures presented by Connelly (1999) or Zhang and Ohsaki (2007):

(1) The member directions do not lie on the same conic at infinity (Connelly, 1999), or equivalently, the geometry matrix of the structure has rank of six for three-dimensional structures (Zhang and Ohsaki, 2007).
(2) $\mathbf{S}$ is positive semi-definite.
(3) $\mathbf{S}$ has maximal rank, which for prismatic tensegrity structures is $6n - 12$.

For prismatic tensegrity structures that are indivisible, the first condition is satisfied, and hence, only the last two conditions need to be considered for verifying super stability of the structures.

When the structure is divisible, the reduced stiffness matrix $\mathbf{Q}$ must have at least one zero eigenvalue, corresponding to the relative motion of the substructures. Thus, divisible structures are not stable.

When the structure is indivisible, and satisfies the third condition, but $\mathbf{S}$ is not positive semi-definite, then the structure may, or may not, be prestress stable. $\mathbf{S}$ must have at least one negative eigenvalue, but whether or not this leads to a negative eigenvalue of $\mathbf{Q}$ depends upon a subtle interplay of the geometrical stiffness matrix and the mechanisms, which themselves depend upon the geometric realisation of the structure.

4.2. Symmetry-adapted forms

Symmetry can be used to simplify calculations and clarify the presentation of the results (Kangwai et al., 1999; Kangwai and Guest, 2000; Ikeda et al., 1992). By using a symmetry-adapted coordinate system, the matrices in a structural calculation can be block-diagonalised. Here, we eventually block-diagonalise the reduced stiffness matrix $\mathbf{Q}$. The block-diagonalisation is simply an orthogonal change of basis, and does not affect the eigenvalues – thus the eigenvalues of $\mathbf{Q}$ are the assembly of the eigenvalues of the individual blocks of the symmetry-adapted $\widetilde{\mathbf{Q}}$.

To block-diagonalise the matrices, we consider symmetry subspaces. Each symmetry subspace corresponds to one of the irreducible representations μ of the group. For the dihedral symmetry group $\mathbf{D}_n$, the irreducible representations are, $A_1, A_2, B_1, B_2, E_1, \ldots, E_{n/2-1}$ for n even, and $A_1, A_2, E_1, \ldots, E_{(n-1)/2}$ for n odd (Bishop, 1973).

The blocks of the symmetry-adapted geometrical stiffness matrix $\widetilde{\mathbf{S}}$ and equilibrium matrix $\widetilde{\mathbf{A}}$ corresponding to μ are denoted by $\widetilde{\mathbf{S}}^{\mu}$ and $\widetilde{\mathbf{A}}^{\mu}$, respectively. The symmetry-adapted mechanisms lying in the null-space of the transpose of $\widetilde{\mathbf{A}}^{\mu}$ are written as columns of $\widetilde{\mathbf{M}}^{\mu}$. The analytical formulations of these symmetry-adapted matrices are presented in Zhang et al. (2009). Then, the block $\widetilde{\mathbf{Q}}^{\mu}$ corresponding to the representation μ of the symmetry-adapted quadratic form $\widetilde{\mathbf{Q}}$ is

$$\widetilde{\mathbf{Q}}^{\mu} = (\widetilde{\mathbf{M}}^{\mu})^{\mathrm{T}} \widetilde{\mathbf{S}}^{\mu} \widetilde{\mathbf{M}}^{\mu}. \tag{29}$$

The matrices $\widetilde{\mathbf{Q}}^{\mu}$ have dimensions of only one or two for prismatic tensegrity structures. The structure is prestress stable if and only if $\widetilde{\mathbf{Q}}^{\mu}$ are positive definite for all representations μ. Note that we have excluded from $\widetilde{\mathbf{Q}}^{\mu}$ the rigid-body motions, which in these cases would correspond to zero eigenvalues of $\widetilde{\mathbf{Q}}^{A_2}$ and $\widetilde{\mathbf{Q}}^{E_1}$.

4.3. Critical factors

Here, we show that the prestress stability of a prismatic tensegrity structure is not only influenced by the connectivity of horizontal cables but also that of the vertical cables, and furthermore, is sensitive to the height/radius ratio. We also show that the selection of materials and level of self-stress is one of the critical factors for the stability of prestress stable structures.

4.3.1. Height/radius ratio

Consider the indivisible structure $\mathbf{D}_7^{3,2}$ in Fig. 9 as an example. The relationship between the minimum eigenvalues of each block $\widetilde{\mathbf{Q}}^{\mu}$ and the height/radius ratio is plotted in Fig. 10.

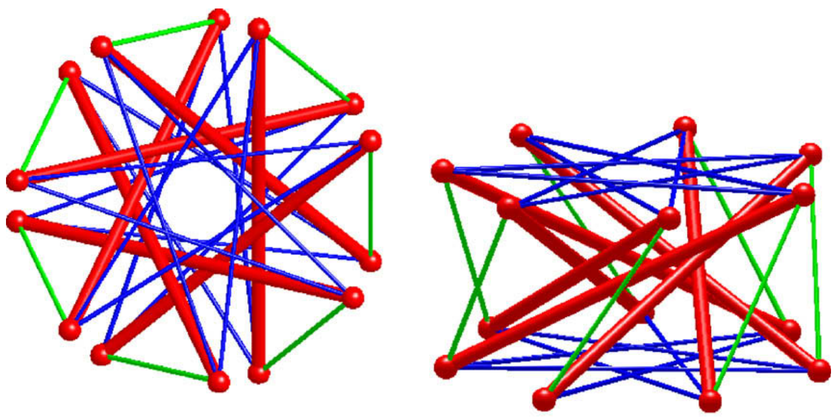
The matrix $\widetilde{\mathbf{Q}}^{A_1}$ is always positive definite, while positive definiteness of $\widetilde{\mathbf{Q}}^{E_2}$ and $\widetilde{\mathbf{Q}}^{E_3}$ vary depending on the height/radius ratio. The structure is prestress stable only when the height/radius ratio falls into the small region [0.75, 1.05], which is shown as a shaded area in the figure.

Consider another indivisible structure $\mathbf{D}_8^{2,3}$ with 16 nodes and 32 members as shown in Fig. 11. The dihedral group $\mathbf{D}_8$ has four one-dimensional and three two-dimensional representations. The relationship of the minimum eigenvalues of $\widetilde{\mathbf{Q}}^{\mu}$ and the height/radius ratio is plotted in Fig. 11. The prestress stability region of the structure ranges from 0.4 to 3.1, which is much wider than that of the structure $\mathbf{D}_7^{3,2}$.

These examples have shown that the height/radius ratio of the structure can be a critical factor in the prestress stability of prismatic tensegrity structures.

4.3.2. Connectivity

As a prismatic tensegrity structure is super stable only if $h = 1$ (Connelly and Terrell, 1995), it is clear that stability of this class of structures is directly related to the connectivity of horizontal cables. It has also been illustrated previously that in some special cases with the right height/radius ratio, the structure can still be prestress stable although it is not super stable. However, this is dependent upon the connectivity of both the horizontal and the vertical cables.



(a) top view (b) side view

Fig. 9. The indivisible structure $\mathbf{D}_7^{3,2}$.

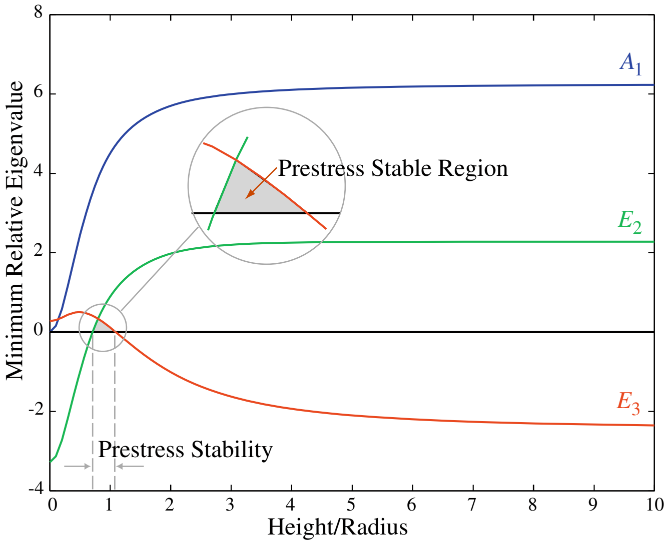


Fig. 10. Influence of the height/radius ratio on the prestress stability of the structure $\mathbf{D}_7^{3,2}$. The structure is prestress stable when the ratio is in the range [0.75, 1.05]. In order to non-dimensionalise the results, the eigenvalues of $\mathbf{Q}$ are plotted relative to the force density in the vertical cables.

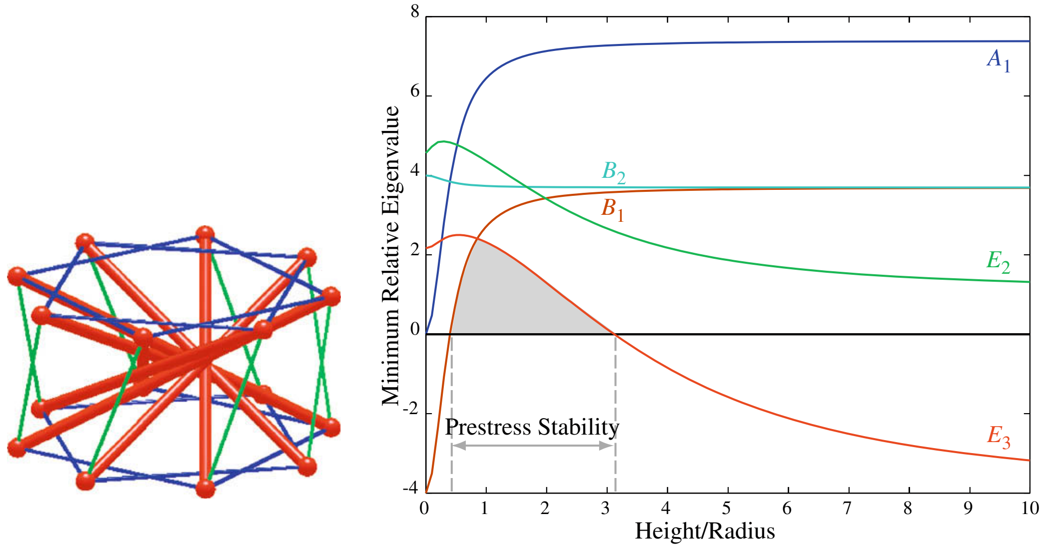


Fig. 11. Influence of the height/radius ratio on the prestress stability of the structure $\mathbf{D}_8^{2,3}$. The structure is prestress stable when the ratio is in the range [0.40, 3.10]. The eigenvalues of $\mathbf{Q}$ are plotted relative to the force density in the vertical cables.

As an example, consider the structures $\mathbf{D}_8^{2,1}$ and $\mathbf{D}_8^{2,3}$, neither of which is super stable, and which only differ in the connectivity of their vertical cables. As we have seen in Fig. 11, $\mathbf{D}_8^{2,3}$ is prestress stable for a limited range of height/radius ratio. By contrast, the structure $\mathbf{D}_8^{2,1}$ in Fig. 12 is never prestress stable, because the minimum eigenvalue of $\widetilde{\mathbf{Q}}^{E_3}$ is always negative.

4.3.3. Materials and self-stresses

So far, the prestress stability is investigated based on the positive definiteness of the quadratic form $\mathbf{Q}$ of the geometrical stiffness matrix with respect to the mechanisms, where the members are assumed to be made of materials with infinite stiffness. Here, we show that selection of materials and level of self-stresses does also affect the stability of the structures when they are not super stable.

We make the simplification that all of the struts and cables have the same axial stiffness. The key parameter is then the ratio of the axial stiffness to the prestress in the structure. Suppose that the cables and struts have axial stiffness AE/l, and that the vertical cables carry a force density of q_v. In the following example, we consider the stiffness for different values of $k = AE/(lq_v)$, where k is dimensionless. If the structure is linear-elastic, the strain due to a particular prestress will be $1/k$, and thus even values of $k = 100$ are too small to be realistic for conventional structures.

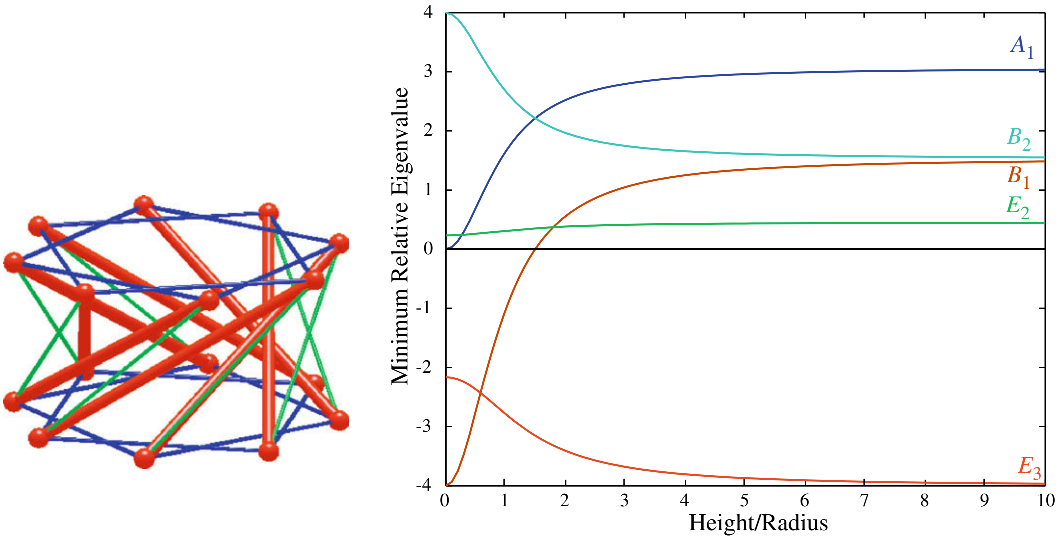


Fig. 12. Influence of the height/radius ratio on the prestress stability of the structure $\mathbf{D}_8^{2,1}$. The structure is never stable. The eigenvalues of $\mathbf{Q}$ are plotted relative to the force density in the vertical cables.

Fig. 13 shows the smallest eigenvalues of the tangent stiffness matrix for the structure $\mathbf{D}_7^{3,2}$, which is prestress stable with the height/radius ratio of 1.0. Results are plotted for $k = 10, 100, 1000$, and for the infinite stiffness case. As k reduces, the structure becomes less stable, and eventually loses stability altogether. Thus, the selection of materials and level of self-stress is also a critical factor to the stability of tensegrity structures.

5. Catalogue

After the stability investigation, we are now in the position to present a catalogue describing the stability of prismatic tensegrity structures for small n:

- $h = 1$: The structures are super stable, and therefore are prestress stable.
- $h \neq 1$: There are two cases:
  - *Divisible*: The structures are divisible, if both of the conditions (22) and (25) are satisfied, and hence, they are not stable,
  - *Indivisible*: Prestress stability can be verified based on the reduced stiffness matrix $\mathbf{Q}$, defined in Eq. (28).

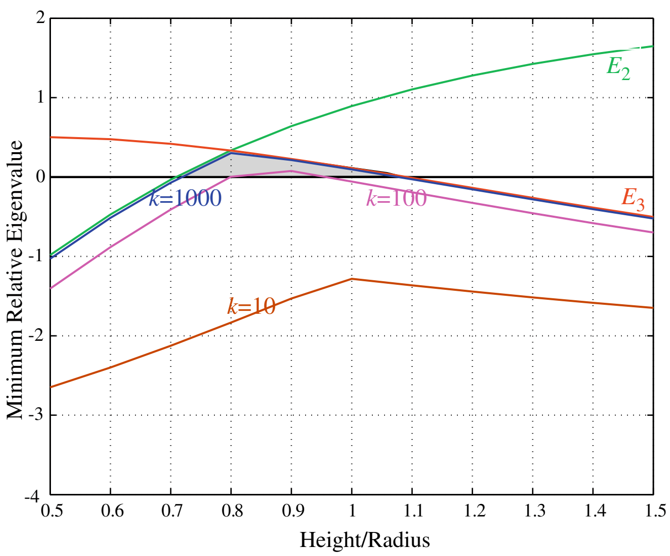


Fig. 13. The influence of the stiffness/self-stress ratio k on the stability of the structure $\mathbf{D}_7^{3,2}$. When k reduces, the structure becomes less stable. The eigenvalues of $\mathbf{K}$ are plotted relative to the force density in the vertical cables.

We present in Table 2 a complete catalogue of prismatic tensegrity structures with symmetry $\mathbf{D}_n$ for $n \leqslant 10$.

From Table 2, it is easy to tell the stability of prismatic tensegrity structures. For example, the structure $\mathbf{D}_6^{2,2}$ can be divided into two identical substructures $\mathbf{D}_3^{1,1}$. Another example: for the structures with $n = 10$ and $h = 2$, the structure $\mathbf{D}_{10}^{2,3}$ is prestress stable in the region [0.70, 1.35], and the structure $\mathbf{D}_{10}^{2,5}$ in Fig. 14 is always prestress stable. Note that all struts of the structure $\mathbf{D}_{10}^{2,5}$ run across the central (origin) point.

6. Discussion and conclusion

A simple symmetry method has been presented to determine the self-equilibrated configuration of a prismatic tensegrity structure with dihedral symmetry. Rather than considering the whole structure, consideration of only one node is sufficient to find force densities and possible configurations.

The necessary and sufficient conditions for the divisibility of prismatic tensegrity structures have been presented based on the connectivity of horizontal and vertical cables. Divisible structures have their own states of self-stresses and rigid-body motions so that they can be physically separated into several identical substructures.

The prestress stability of prismatic tensegrity structures is demonstrated to be related to the connectivity of the cables, and is also sensitive to the height/radius ratio. It is also shown that stability of a tensegrity structure that is not super stable is influenced by the selection of materials and level of self-stress.

Table 2
The stability of prismatic tensegrity structures $\mathbf{D}_n^{h,v}$

		h
$n = 3$		1
v	1	s

		h	
$n = 4$		1	2
v	1	s	n
	2	s	$2\mathbf{D}_2^{1,1}$

		h	
$n = 5$		1	2
v	1	s	n
	2	s	n

		h		
$n = 6$		1	2	3
	1	s	n	n
v	2	s	$2\mathbf{D}_3^{1,1}$	n
	3	s	p	$3\mathbf{D}_2^{1,1}$

		h		
$n = 7$		1	2	3
	1	s	n	n
v	2	s	n	[0.75,1.05]
	3	s	n	n

		h			
$n = 8$		1	2	3	4
	1	s	n	n	n
v	2	s	$2\mathbf{D}_4^{1,1}$	n	$2\mathbf{D}_4^{2,1}$
	3	s	[0.40,3.10]	n	n
	4	s	$2\mathbf{D}_4^{1,2}$	[0.35,2.35]	$4\mathbf{D}_2^{1,1}$

		h			
$n = 9$		1	2	3	4
	1	s	n	n	n
v	2	s	n	n	n
	3	s	n	$3\mathbf{D}_3^{1,1}$	n
	4	s	n	[0.20,1.60]	n

		h				
$n = 10$		1	2	3	4	
	1	s	n	n	n	n
	2	s	$2\mathbf{D}_5^{1,1}$	n	$2\mathbf{D}_5^{2,1}$	n
v	3	s	[0.70,1.35]	n	[0.75,1.25]	n
	4	s	$2\mathbf{D}_5^{1,2}$	n	$2\mathbf{D}_5^{2,2}$	n
	5	s	p	p	p	$5\mathbf{D}_2^{1,1}$

's' denotes super stable, 'n' denotes not stable, and 'p' indicates that the structure is not super stable but is always prestress stable with arbitrary height/radius ratio. If the structure is prestress stable only in a specific region of height/radius ratio from h_1 to h_2, then this region is given by $[h_1, h_2]$; and if the structure is divisible, its substructures are given.

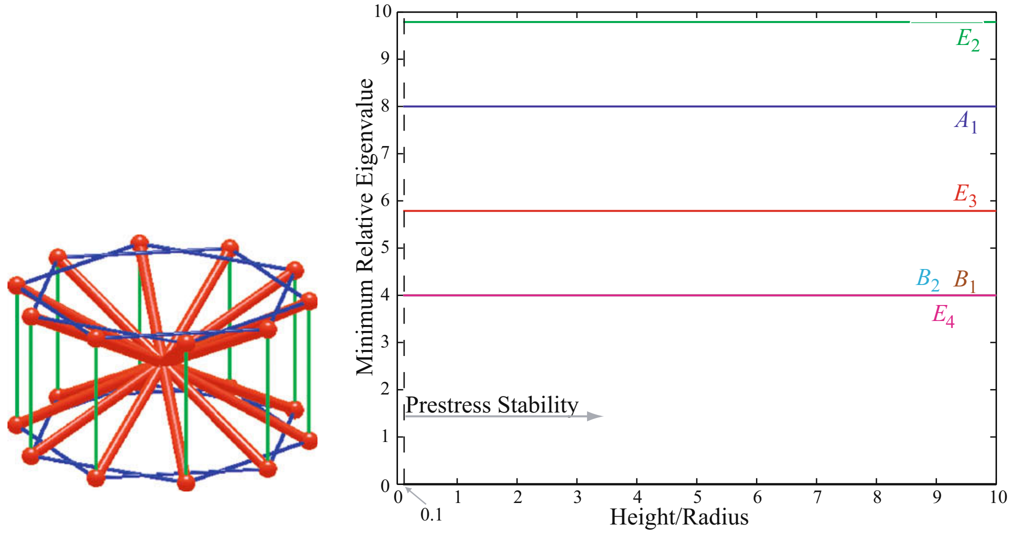


Fig. 14. The structure $\mathbf{D}_{10}^{2,5}$ that is not super stable but is always prestress stable.

A catalogue of the prismatic tensegrity structures with relative small number of members has been presented. We have also developed a Java program to enable designers to interactively design the prismatic tensegrity structures. The program is published online: http://tensegrity.AIStructure.com/prismatic/

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